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PLANE ASTRONOMY.

THE ELEMENTS

S.H. 1827.

OF

The Theory

OF

PLANE ASTRONOMY.

BY

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ADVERTISEMENT.

The following compendium is designed to embrace, in a short compass, demonstrations of the chief mathematical propositions of Plane Astronomy, and is principally intended for those readers who may not have the inclination or opportunity to study the science with reference to its practical application. In pursuance of this design, all allusion to the business of an observatory, to the history of the science, and to the brilliant speculations of modern Astronomers have been avoided. As this mode of treating the subject must deprive it of much that is interesting, and also, if it be considered independent of the general end of mathematical studies, of much that is useful, it is probable that few students will be contented with the slight portion of astronomical knowledge contained in this work. To those who would know more than the mere theory of Astronomy, the valuable treatise of Professor Woodhouse is indispensible. Much assistance may also be obtained from the works of Biot, Delambre and Littrow, from which, (especially the latter), the Author has derived very considerable assistance.

The Author begs leave to express his grateful acknowledgments to the Syndics of the University Press, for their liberal contribution to the expence of publishing this book.

St. John's College,
Oct. 26, 1826.

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PLANE ASTRONOMY.

CHAP. I.

INTRODUCTION.

1. **Astronomy** is that branch of Natural Philosophy, which treats of the real and apparent motions of the heavenly bodies. In *Plane Astronomy,* both of these are investigated from observation and calculation, and in *Physical Astronomy,* the former are accounted for on mechanical principles.

The starry heavens appear, to a spectator on the Earth's surface, to be a vast spherical vault, of which his eye forms the center, and in which the heavenly bodies are situated. It is certain that this appearance is fallacious, and that the stars are not all at the same distance from us, but that the eye, being unable to judge of the distance of very remote bodies, and having no scale to compare it with, supposes them all equally distant.

2. Some of these bodies seem never to change their relative situation, whence they have been called *Fixed Stars;* the angle which any two of them subtend, is the same in all parts of the Earth, from which it is evident, that the distance of any two points on the Earth's surface is evanescent, compared with the distance of the fixed stars from the Earth; therefore, instead of supposing the center of the superficies in which they appear to lie, to be the eye of the spectator, we may suppose it to be the center of the Earth.

3. If the plumb-line be produced indefinitely both ways, it meets the celestial sphere in two points, of which, that

above our heads is called the *Zenith*, and the opposite one the *Nadir*. A plane perpendicular to this line at the surface of the Earth, is called the *Sensible Horizon*, and determines the visible portion of the heavens; a plane parallel to the former, drawn through the Earth's center, is called the *Rational Horizon;* as the radius of the Earth subtends no perceptible angle at the distance of the stars, the circles in which the celestial vault is cut by these planes, may be supposed to coincide, and any errors, introduced by this supposition, with respect to bodies at which the Earth's radius subtends a finite angle, may be afterwards taken into account and corrected.

4. If the heavenly bodies are attentively observed, they appear to describe circles, in planes parallel to each other, about a common axis, which meets the heavens in two points called the *North* and *South Poles*. To an inhabitant of the northern hemisphere, the stars situated near the north pole, which are called *Circumpolar Stars*, never set, but continue visible during the whole of their circular course, others more to the south, ascend from the east side of the horizon, and after having attained their greatest height, which is called *Culminating*, descend, and disappear in the west, completing only part of a circle above the horizon. To an inhabitant of the southern hemisphere, the heavens present a spectacle the reverse of this; the stars near the south pole are seen to describe complete circles, whilst those more to the north, are only seen in part of their course, or if near the north pole, never rise above the horizon.

5. The parallel circular motion of the heavens may be observed by an instrument, such as is represented in fig. 1, consisting of a circle RQT moveable round an axis NS, which coincides with one of its diameters. To the plane of the circle is attached a telescope TQ, moveable round C in the plane RQT. If the axis NS be fixed in a proper position, which may be done in a few trials, and the telescope be pointed towards a fixed star, it will be found, that the star may be seen through the telescope, as long as it continues above the horizon, by making the circle revolve, without disturbing the place of the

telescope in the plane RQT; therefore as the telescope describes a conical surface round NS, the curve in which it cuts the celestial sphere whose center is C, or the path which the star S describes, is a circle. Any other star may be observed in its course, by changing the position of the telescope, without altering the position of NS; therefore all the stars describe circles in planes perpendicular to NS, and having their centers situated in that line.

The angular motion round NS is uniform, and is the same for all stars; for if the telescope be made to point to a star on one day, and be left in that position, it will point to the same, star on the next day, after a certain interval, which is the same whatever star is chosen, and at whatever time the first observation is made.

6. The great circle of the celestial sphere, whose plane is parallel to the planes in which the stars move, or perpendicular to the axis passing through the north and south poles, is called the *Equinoctial* or *Equator.*

Declination circles are great circles whose planes pass through the north and south poles; they are perpendicular to the equator and are sometimes called *Meridians.* The *declination* of a star is its angular distance from the equator, measured on the arc of a declination circle, intercepted between the star and the equator. The polar distance is the angular distance from the pole, or the complement of the declination.

The *Meridian* of a place on the Earth's surface is the declination circle passing through the zenith of the place; it is perpendicular to the horizon, which it intersects in a line called a *Meridian Line.*

The angle between the meridian of a place and a declination circle passing through a star, is called the star's *Hour Angle,* because the stars revolving uniformly round the Earth's axis, the angle between these circles varies as the time of a star's moving from one to the other.

7. The stars appear to revolve round the Earth's axis, or to describe an angle of 360^0, in an interval of $23^h.\ 56'$, which is divided into 24 equal parts, called *Sidereal Hours,* and therefore move over 15^0 in one sidereal hour.

8. *Vertical Circles* are circles whose planes pass through the zenith and nadir; of these the *Prime Vertical* is perpendicular to the meridian, and cuts the horizon in the *east* and *west* points.

Since the equator and prime vertical are each perpendicular to the meridian, their intersection is perpendicular to that plane, and the same is true of the intersection of the prime vertical and horizon; hence, these three planes intersect in the same line.

The *Altitude* and *Zenith Distance* of a star are the respective distances from the horizon and zenith measured on the arc of a vertical circle. Hence the altitude together with the zenith distance of a star $= 90^0$.

The *Azimuth* of a star, is the angle between a vertical circle passing through it, and the meridian of the place.

The section of the Earth's mean surface, made by the plane of the meridian, is called the *Terrestrial Meridian,* and that made by the plane of the equator, is called the *Terrestrial Equator.*

9. The *Latitude* of a place on the Earth's surface is the angular distance of its zenith from the equator, measured by the intercepted arc of the meridian; the complement of this, or the distance from the pole is the *Co-Latitude.*

Since the latitude + the zenith distance of the pole $= 90^0 =$ the altitude of the pole + the zenith distance of the pole; therefore the latitude of any place is equal to the altitude of the pole at that place.

10. Besides the *Diurnal Motion,* already spoken of, common

to all the heavenly bodies, some of them have *proper* motions, and are seen to change their place with respect to the fixed stars. The most remarkable bodies which have *proper* motions are the Sun and Moon; the former of these moves in a great circle called the *Ecliptic*, and after having passed round the heavens, returns to the same position at the end of a year. The inclination of the ecliptic and equator is the *Obliquity*, and the line of intersection meets the celestial sphere in two points called the *Vernal* and *Autumnal Equinoxes*. The ecliptic is divided into twelve equal parts or *Signs*, each consisting of 30^0, and denoted thus; Aries (♈), Taurus (♉), Gemini (♊), Cancer (♋), Leo (♌), Virgo (♍), Libra (♎), Scorpio (♏), Sagittarius (♐), Capricornius (♑), Aquarius (♒), Pisces (♓). The *first point of Aries* is the vernal equinox, where the Sun appears in northern latitudes at the beginning of spring, or when it passes from the south to the north side of the equator.

11. The *Planets*, which are distinguished from the fixed stars by their apparent magnitudes, have also proper motions, and move round the Sun in different periods, in orbits nearly circular.

The planets were supposed to be confined to a space of 8^0 on each side of the ecliptic, which was called the *Zodiac*, and on this account the divisions of the ecliptic were called *Signs of the Zodiac*.

Comets are bodies which move round the Sun in very eccentric orbits, when first seen, the body of the comet, called the *Nucleus*, is generally followed by a train of faint light called the *Tail*, which becomes brighter as the comet approaches the Sun. After that time, the comet diminishes in brilliancy, and by its very rapid motion, is soon carried into the distant regions of space, and becomes invisible to the Earth.

12. The *Solstitial Colure* is a great circle passing through the poles of the equator and ecliptic, and intersecting the ecliptic in two points called the *Solstices*. Since this circle passes through the poles of the equator and ecliptic, it is per-

pendicular to each of them, and, therefore, to the line of their intersection, and therefore, cuts them at 90° from the equinoxes. The solstice which lies north of the equator, and in which the Sun appears at Midsummer, is called the *Summer* Solstice: the other, in which the Sun is at the beginning of Winter, is called the *Winter* Solstice. The *Equinoctial Colure* is a declination circle passing through the equinoxes.

The *Latitude* of a Star is its angular distance from the ecliptic. The *Longitude,* is its distance from the first point of Aries, measured along the ecliptic, and the *Right Ascension* the distance from the same point, measured along the equator; the Star being referred to those planes by *Secondaries* to the ecliptic and equator, that is, by circles passing through the respective poles of those circles.

The angle contained between the circles of declination and latitude, passing through a Star, is called its *Angle of Position.*

The place of a body in the celestial hemisphere is determined by its latitude and longitude, or by its right ascension and declination, just as the position of a point in a plane is determined by its two co-ordinates. Latitude and declination are measured north and south through 90°; right ascension and longitude from west to east, or in a direction opposite to that in which the stars appear to move, through 360°.

The *Terrestrial Longitude* of a place on the Earth's surface is the angle between the meridian of the place and another called a *First Meridian,* passing through a given place; it is measured east and west from the first meridian, along the equator through 180°. The latitude and longitude of a place, on the Earth's surface, must be carefully distinguished from the latitude and longitude of a star.

The stars are divided into groups called *Constellations,* and are distinguished by the Greek letters α, β, γ, &c. in the order of the star's apparent magnitude; thus γ Draconis is a star of the third magnitude in the constellation Draco. By magnitude here is meant brilliancy, for the stars observed with telescopes of the strongest magnifying powers have no apparent diameters, the different magnitudes which they appear to have to the naked eye, or in inferior telescopes, being an optical illusion, caused by the radiation of light.

13. The preceding definitions may be readily understood, by observing the circles on a common celestial globe. They are drawn in fig. 2, which represents the western hemisphere, Z is the *zenith*, or pole of the *horizon* HR, P the pole of the equator TQ, K of the ecliptic C♈k, H, W, R, are the *North*, *West*, and *South* points of the horizon respectively. HZR the *meridian*, and ZQ or PH the *latitude* of a place, the zenith of which is Z. Through a star S, draw the circles ZSz, PSp, KSk; then ZS is the *zenith distance* of S, zS the *altitude*, PZS or Hz the north azimuth, ZPS or Qp the *hour angle*, PS the *polar distance*, pS the *declination*, Sk the *latitude*, ♈k the *longitude*, ♈p the *right ascension*, and KSP the *angle of position*; k♈p is the *obliquity*. The relation between these quantities is calculated by Spherical Trigonometry.

14. Altitudes, zenith distances, and the angular distance between two bodies, or two parts of the same body, can be *observed* with proper instruments. In an Observatory the best graduated instruments are fixed in the plane of the meridian, and the altitudes observed with them are meridian altitudes, which are those used in the most important calculations. Azimuths can also be observed, but not with the same accuracy as altitudes, and are, therefore, not often resorted to. The difference of right ascensions of two fixed stars, may also be considered a result of observation, for, if the times of their transit over the meridian be observed by the sidereal clock, 1^h : the observed interval :: 15^0 ; difference of right ascension [*].

15. It has been said that the stars appear to describe parallel circles round the axis of the equator in $23^h.56'$, but

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[*] See Brinkley's *Astronomy*, Chap. XI, Woodhouse's *Astronomy*, Chap. V.

it is evident that the same effect will be produced if the stars remain at rest, and the Earth revolve uniformly round the same axis, in the same time, but in the opposite direction. For the spectator, not being aware of his own motion, would refer the whole change of relative position to the stars which would appear to describe parallel circles in a direction opposite to his own motion. That the Earth really revolves is rendered probable; 1st, by its greater simplicity, when we consider the immense distance of the stars, and the enormous rapidity with which they must revolve, in order to move round the Earth in $23^h.56'$; 2d, from the analogy of the Sun and planets, which, although many of them much larger than the Earth, are found to have such a rotation; 3d, from the Earth's being not exactly spherical, but nearly such as it would be, supposing it to have been originally fluid, and to have revolved round an axis; and from the diminution of gravity in proceeding towards the equator being such as would arise from the centrifugal force of the Earth's rotation. It has been tried to determine the question experimentally, from considering that if the Earth revolve round an axis, a stone let fall from the top of a tower, ought not to fall exactly at the foot, but a little to the east of it; for the stone, at the moment of falling, has the horizontal velocity of the top of the tower, and, as this is greater than that of the places it successively arrives at, it ought to have a relative easterly motion from the tower, and ought, therefore, to fall east of the foot of the tower. The experiments have not been very decisive, on account of the difficulty of measuring accurately the small deflection from the vertical, but they seem to favor the hypothesis of the Earth's rotation.

16. The same difficulty of distinguishing between real and apparent motion occurs with regard to the Sun. The interval between the setting of the Sun and a star which sets after it diminishes each day, till the star is obscured by the Sun's light; a few days afterwards, the star rises a little before the Sun, and the interval goes on increasing, whence the Sun seems to move through the heavens in a direction opposite to the apparent diurnal motion, or from west to east; but the Sun would

appear to move in the same way if it remained at rest, and the Earth moved round it in the same direction and period. For, let S (fig. 3.) be the Sun, E, E' the earth, in any two points of its orbit round S; through E' draw $E's$ parallel and equal to ES. Then, when the Earth was at E, the Sun appeared among the fixed stars in ES produced, or on account of their immense distance in $E's$ produced, and when the Earth is at E', the Sun is seen in the direction $E'S$, therefore, it has appeared to describe the angle $SE's$ round the Earth, $=ESE'$, the angle described by the Earth round the Sun. The Sun's path on the surface of the sphere is circular, and depends only on the plane in which it moves, but since when the Earth was at E, the distance of the Sun was $ES=E's$, and that the angle $ESE'=SE's$, therefore, the Sun must appear to describe the same curve round the Earth which the Earth describes round the Sun.

17. There is reason to suppose that the Earth really moves round the Sun. For, from the distance of the Earth from the Sun, and the period of the Sun's apparent motion round the Earth, compared with that of the planets, which we know revolve round the Sun, it seems probable that the Earth is a planet, and is subject to the same laws of motion as the others. The aberration of light also, and the stationary points of the superior planets, which will be treated of hereafter, can be simply explained on the hypothesis of the Earth's motion round the Sun, and have not been accounted for on the contrary supposition, We conclude, therefore, that the Earth revolves round its axis in a day, and round the Sun in a year, but as the general phenomena would be the same on the contrary hypothesis, we may, in explaining them, make use of that supposition, which is most convenient.

CHAP. II.

ON THE FIGURE OF THE EARTH.

18. By the Figure of the Earth, is not meant the Earth with all its inequalities, but an imaginary figure, such as would be found by reducing hills and valleys to the same level, or such a figure as the surface of the sea presents us. The Earth, at first sight, appears to one of its inhabitants, to be a vast plane extended infinitely on all sides, but the following considerations, from which its round figure may be inferred, are so obvious, that this inference must have been made very early. When a ship approaches land, the topmast first becomes visible to a spectator on the shore, and the hull comes in sight last, which being the largest part, ought to be first seen, if the Earth were a plane. In advancing towards the north, the pole star seems to approach the zenith, whilst the contrary effect takes place in moving towards the South. In lunar eclipses, when the Earth is interposed between the Sun and Moon, the Earth's shadow upon the Moon's disk appears round in all positions of the Earth, and, lastly, the Earth has been sailed round in various directions.

19. The simplest of round figures is a sphere, and the Earth was supposed to be such, after its roundness had been determined by observations similar to the above, but the analogy of some of the planets which seem to be considerably depressed at their poles, together with considerations on the form which a revolving fluid would put on, lead to the conclusion that the Earth is more nearly an oblate spheroid, having its axis of revolution perpendicular to the equator. It is possible to put these

hypotheses to the test of experiment, by measuring arcs of the meridian in different latitudes, and, as the knowledge of the true figure of the Earth is of much importance, this has been undertaken in various parts of the globe, and large arcs have been measured with great accuracy in England, France, Lapland, Sweden and India. The whole distance between the extremities of the arc of the meridian to be determined, is not actually measured, which would in all cases be laborious, and in most, impossible. A base of four or five miles is first measured, with the utmost care, which is connected with the two extremities of the arc of the meridian by means of triangles, and the angles of these triangles being observed, and the sides calculated by Trigonometry, the length of the arc of the meridian may be ascertained. These operations are of considerable difficulty, and have been carried on by the ablest modern astronomers, furnished with the best instruments, and to their works* the reader is referred for information on this subject, we shall only shew here, how the degrees of the meridian, when measured, may be used in ascertaining the figure of the Earth.

20. *The difference of the meridian zenith distances of the same star is equal to the difference of latitude of the places of observation.*

Let pe (fig. 4.) be a terrestrial meridian; Ce, Cp, the equatoreal and polar radii, ZAG, $Z'A'G'$, normals at the points AA', meeting in O; therefore, AGE (l) and $A'G'e$ (l') are the latitudes of A and A'. Let ZAS (z) and $Z'A'S$ (z') be the zenith distance of the same star; therefore, on account of the great distance of the fixed stars, AS is parallel to $A'S$;

$$\begin{aligned} \therefore\ z' = Z'A'S' &= z + A'OA \\ &= z + (l' - l); \\ \therefore\ z' - z &= l' - l \text{ or } A'OA. \end{aligned}$$

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* *Philosophical Transactions* for 1787, 1790, 1803, 1818. Delambre, *Base du Système Metrique.*

21. Cor. 1. Hence, by observing at what places, on the same meridian, the difference of the meridian zenith distances of a star is one degree, and measuring the arc between them, the length of an arc of one degree is determined.

22. Cor. 2. If the terrestrial meridian were circular, O would coincide with C, and the angle of one degree would always be subtended by the same arc, but the length of a degree is found to increase as we advance from the equator towards the pole, which shews that the curvature diminishes from the equator to the pole.

23. *To find the radius of curvature at any point of the terrestrial meridian, supposing the Earth to be an oblate spheroid.*

Let pe (fig. 5.) be the elliptic meridian; Ce (a) the equatoreal, Cp (b) the polar radius; PG (n) a normal to any point P, meeting the semi-conjugate in F, and Cp in R. Let $CN=x$, $PN=y$, $PGN=l$, the eccentricity $=e$, and the difference $a-b=c$.

$$\text{Then } n^2 = PN^2 + GN^2$$

$$= y^2 + \frac{b^4}{a^4} \cdot x^2$$

$$= y^2 + \frac{b^4}{a^2} \cdot \left(1 - \frac{y^2}{b^2}\right)$$

$$= y^2 \cdot \left(1 - \frac{b^2}{a^2}\right) + \frac{b^4}{a^2}.$$

$$= n^2 \sin^2 l \,.\, e^2 + \frac{b^4}{a^2};$$

$$\therefore\ n^2 = \frac{b^4}{a^2} \cdot \frac{1}{1 - e^2 \sin^2 l};$$

$$\therefore\ n = a \cdot \frac{1 - e^2}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}.$$

But the radius of curvature $(r) = \dfrac{a^2 b^2}{PF^3}$

$$= \frac{a^2 b^2 . n^3}{b^6} = \frac{a^2 n^3}{b^4};$$

$$\therefore \; r = \frac{a^2}{b^4} \cdot \frac{b^6}{a^3} \cdot \frac{1}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}}$$

$$= \frac{b^2}{a . (1 - e^2 \sin^2 l)^{\frac{3}{2}}}$$

$$= a . (1 - e^2) . (1 - e^2 \sin^2 l)^{-\frac{3}{2}}$$

$$= a . (1 - e^2) . \left(1 + \frac{3}{2} e^2 \sin^2 l\right),$$

neglecting e^4, &c.

$$= a . \left(1 - e^2 + \frac{3}{2} e^2 \sin^2 l\right).$$

But $e^2 = \dfrac{a^2 - b^2}{a^2} = \dfrac{a^2 - \overline{a - c}|^2}{a^2} = \dfrac{2c}{a}$ nearly;

$$\therefore \; r = a - 2c + 3c \sin^2 l.$$

24. Cor. $y = n \sin l = \dfrac{a . (1 - e^2)}{\sqrt{1 - e^2 \sin^2 l}} . \sin l,$

$$x = \frac{a^2}{b^2} . AG = \frac{a^2}{b^2} . n \cos l$$

$$= \frac{a \cos l}{\sqrt{1 - e^2 \sin^2 l}},$$

$$CP^2 = x^2 + y^2 = a^2 . \frac{\overline{1 - e^2}|^2 . \sin^2 l + \cos^2 l}{1 - e^2 \sin^2 l}$$

$$= a^2 \{(1 - 2 e^2 \sin^2 l) . (1 + e^2 \sin^2 l) \text{ nearly}$$

$$= a^2 (1 - e^2 \sin^2 l);$$

$$\therefore CP = a.\{1 - \tfrac{1}{2}e^2 \sin^2 l\}$$
$$= a - c \sin^2 l.$$

25. Since a spheroid is a solid of revolution, the normal to the surface at any point will pass through the axis; if, therefore, *PR*. *QR*, (fig. 5.) be two consecutive normals to the very small arc *PQ*, which is perpendicular to the meridian, they will cut the axis in the same point *R*; $\therefore$ *PR* (*R*) is the radius of curvature of *PQ*.

$$\text{And } R = PG.\frac{CN}{PN} = n.\frac{a^2}{b^2}$$
$$= \frac{a}{\sqrt{1 - e^2 \sin^2 l}}$$
$$= a(1 + \tfrac{1}{2}e^2 \sin^2 l)$$
$$= a + c \sin^2 l.$$

Cor. Since $2c \sin^2 l$ is less than $2c$; therefore $a - 2c + 3c \sin^2 l$ is less than $a + c \sin^2 l$; and, therefore, the curvature of the terrestrial meridian, is greater than that of the perpendicular arc.

26. Since the parts of the Earth's surface on each side of the meridian are similar, upon the hypothesis of the Earth's being a solid revolution; therefore of all sections of the surface, made by planes passing through the normal, the meridian must be the section of greatest curvature, and if ρ be the radius of curvature of a section made by a plane passing through the normal inclined at an angle a to the meridian,

$$\rho = \frac{Rr}{R \cos^2 a + r \sin^2 a}.*$$

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* To deduce this expression for the radius of curvature of a curve on a surface, passing through a given point, take the tangent plane at the point for the plane of xy, and the normal for the axis of z, and let the plane

$$= a + c \sin^2 l - 2c \cos^2 a \,.\, \cos^2 l,$$

substituting the values of R and r, and reducing.

---

plane passing through the normal and curve be inclined at an angle a to the plane of xz. Then, because the radius of a circle $= \frac{1}{2} \cdot \frac{\text{chord}^2}{\text{versin}}$; therefore the radius of the circle which coincides with the curve at the origin $(\rho) = \frac{1}{2} \cdot \frac{x^2+y^2+z^2}{z}$, when x, y, z vanish.

To find the value of this fraction, let y', z', z'' denote the differential coefficients taken with regard to x, and $z_{,}$, $z_{,,}$ with regard to y, and observe that y is a function of x given by the equation $y = x \tan a$.

$$\text{Then } \rho = \frac{1}{2} \cdot \frac{x^2+y^2}{z}$$

$$= \frac{x + yy'}{z' + z_{,}y'}$$

$$= \frac{1 + y'^2 + yy''}{z'' + 2z_{,}'y' + z_{,,}y'^2 + z_{,}y''}$$

$$= \frac{1 + \tan^2 a}{z'' + 2z_{,}' \tan a + z_{,,} \tan^2 a}.$$

But because r is the minimum radius of curvature of all curves passing through the given point, $a = 0$ ought to satisfy the equation $\frac{d\rho}{da} = 0$.

$$\text{Or } 0 = \tan a \,.\, (z'' + 2z_{,}' \tan a + z_{,,} \tan^2 a) - (1 + \tan^2 a) \,.\, (z_{,}' + z_{,,} \tan a).$$

$$\text{Or } z_{,}' \tan^2 a + (z'' - z_{,,}) \,.\, \tan a - z_{,}' = 0;$$

$$\therefore z_{,}' \text{ must equal } 0.$$

$$\therefore \rho = \frac{1 + \tan^2 a}{z'' + z_{,,} \tan^2 a};$$

$$\therefore r = \frac{1}{z''}, \text{ making } a = 0,$$

$$R = \frac{1}{z_{,,}}, \text{ making } a = 90^\circ;$$

$$\therefore \rho = \frac{Rr}{R \cos^2 a + r \sin^2 a}.$$

27. Having given D and D', the lengths of two degrees of the meridian whose middle points are in latitudes l and l', to find a, c, and $\frac{c}{a}$.

If m $(=57^0.295)$, be the number of degrees in an arc=radius,

$$mD = a - 2c + 3c \sin^2 l,$$

$$mD' = a - 2c + 3c \sin^2 l';$$

$$\therefore m.(D - D') = 3c.(\sin^2 l - \sin^2 l');$$

$$\therefore c = \frac{m.(D - D')}{3(\sin^2 l - \sin^2 l')} = \frac{m(D - D')}{3.\sin(l + l').\sin(l - l')}$$

and, substituting this value in the first equation,

$$a = m.\frac{2(D - D') + 3(D' \sin^2 l - D \sin^2 l')}{3 \sin(l + l').\sin(l - l')};$$

therefore, by division,

$$\frac{c}{a} = \frac{D - D'}{2(D - D') + 3(D' \sin^2 l - D \sin^2 l')}$$

$$= \frac{D - D'}{3(D' \sin^2 l - D \sin^2 l')},$$

neglecting the square of $D - D'$.

The quantity $\frac{c}{a}$ is called the compression.

28. Cor. 1. If E be the degree at the equator, P at the pole, and M at the mean latitude 45^0,

$$mE = a - 2c,$$

$$mP = a + c,$$

$$mM = a - \tfrac{1}{2}c;$$

$$\therefore M = \tfrac{1}{2}(P + E).$$

29. Cor. 2. The excess of a degree in any latitude above that at the equator,

$$\text{or } D - E \propto \sin^2 l.$$

30. Given D, the length of a degree of the meridian, and Δ the length of a degree perpendicular to the meridian; to find a and c.

$$m\Delta = a + c \sin^2 l,$$

$$mD = a - 2c + 3c \sin^2 l;$$

$$\therefore m(\Delta - D) = 2c \cos^2 l,$$

$$\therefore c = \frac{m.\overline{\Delta - D}}{2 \cos^2 l},$$

$$a = m.\{\Delta - \tfrac{1}{2}.(\Delta - D).\tan^2 l\};$$

$$\therefore \frac{c}{a} = \tfrac{1}{2}.\frac{\Delta - D}{\Delta.\cos^2 l},$$

neglecting the square of $\Delta - D$.

In the same manner, the compression of the terrestrial spheroid may be found, by measuring degrees on arcs inclined at any given angles to the meridian, by means of the formula

$$p = a + c \sin^2 l - 2c \cos^2 a . \cos^2 l.$$

31. *From any two measured arcs of the meridian, to determine a and c.*

Let $Pc = s$, and m, r, l as before.

$$\text{Then } mds = rdl$$

$$= (a - 2c + 3c \sin^2 l).dl$$

$$= (a - \frac{1}{2}c - \frac{3}{2}c \cos 2l)\, dl;$$

$$\therefore ms = al - \tfrac{1}{2}cl - \tfrac{3}{4}.c \sin 2l + \text{corr.}$$

Let S be the whole arc taken between the values of l, l and l';

$$\therefore\ mS = a.(l'-l) - \tfrac{1}{2}c.(l'-l) - \tfrac{3}{4}.c(\sin 2l' - \sin 2l)$$

$$= pa + qc,$$

where $p = l' - l$,

and $q = -\frac{1}{2}.(l'-l) - \frac{3}{4}.(\sin 2l' - \sin 2l)$.

Similarly for another arc S',

$$mS' = p'a + q'c;$$

$$\therefore\ a = m.\frac{q'S - qS'}{pq' - q'p},$$

$$c = m.\frac{p'S - pS'}{p'q - q'p}.$$

32. The approximate values of a, b and c, have been deduced in the preceding articles, because they are more simple than the exact values, and because, from the nearly spherical figure of the Earth, the errors thus introduced may be safely neglected. The exact values may, however, be obtained with nearly equal facility, and from these, although not so easily, the approximate values, which are better adapted for numerical calculation, may be deduced by neglecting powers of e above the second, or powers of $D - D'$ above the first. The exact values are as follows:

$$r = \frac{a^2b^2}{(a^2\cos^2 l + b^2\sin^2 l)^{\frac{3}{2}}},$$

$$R = \frac{a^2}{(a^2\cos^2 l + b^2\sin^2 l)^{\frac{1}{2}}},$$

$$a = \frac{mDD'.\sin^{\frac{3}{2}}(l+l').\sin^{\frac{3}{2}}(l-l')}{(D'^{\frac{2}{3}}.\cos^2 l' - D^{\frac{2}{3}}.\cos^2 l).(D^{\frac{2}{3}}\sin^2 l - D'^{\frac{2}{3}}.\sin^2 l')^{\frac{1}{2}}},$$

$$b = \frac{mDD'.\sin^{\frac{3}{2}}.(l+l').\sin^{\frac{3}{2}}(l-l')}{(D^{\frac{2}{3}}\sin^2 l - D'^{\frac{2}{3}}\sin^2 l')(D'^{\frac{2}{3}}\cos^2 l' - D^{\frac{2}{3}}\cos^2 l)^{\frac{1}{2}}},$$

$$e^2 = \frac{D^{\frac{2}{3}} - D'^{\frac{2}{3}}}{D^{\frac{2}{3}} \sin^2 l - D'^{\frac{2}{3}} \sin^2 l'},$$

$$a = \frac{m \Delta^{\frac{3}{2}} \cos l}{\sqrt{\Delta - D \sin^2 l}},$$

$$b = \frac{m D^{\frac{1}{2}} . \Delta^{\frac{3}{2}} \cos^2 l}{\Delta - D \sin^2 l},$$

$$e^2 = \frac{\Delta - D}{\Delta - D \sin^2 l}.$$

33. In applying the above formulæ to the determination of the compression, from the length of two degrees of the meridian in different latitudes, those arcs ought to be chosen, which differ most from each other in situation, as being more likely to cause a compensation of errors. For although the general figure of the meridian, from the pole to the equator may be an ellipse, yet particular parts may differ considerably from it. The pairs of degrees best adapted for this purpose are, those in France and Peru, France and Bengal, France and Lapland, Lapland and Bengal; which give the respective compressions $\frac{1}{310.6}$, $\frac{1}{311.3}$, $\frac{1}{315.2}$, $\frac{1}{312.9}$. These coincide sufficiently with each other to prove the truth of the hypothesis of the Earth's spheroidal figure, and the compression may be considered equal to the mean of these $\frac{1}{312.5}$. With this compression,

Miles.

$a = 3962.349$,

$b = 3949.669$,

$c = 12.680$.

The latitude of the extremities of an arc of the meridian cannot be depended on to more than 1″, and the deviation of the plumb-line, caused by the attraction of mountains, may sometimes cause greater errors. From this cause, or from real inequalities, the meridian in some places differs considerably from the meridian determined above. The compression calculated from the arc passing through France is as great as $\frac{1}{213}$; and the degree at the Cape of Good Hope, measured by Lacaille in south latitude 37^0, is greater than that in Pensylvania in north latitude 43^0, which, if there is no error in the measurement, would shew that the southern and northern hemispheres are unequal.

34. The definition of the terrestrial meridian, given in the preceding chapter, is not quite correct, for unless the Earth be a solid of revolution, the meridian will not be a plane curve. The celestial meridian passes through the axis of the Earth and through the zenith of the observer, and all places of the Earth, which have their zeniths in the same celestial meridian, may be considered as having the same terrestrial meridian. Now, considering the immense distance of the stars, the vertical at each of these places may be considered parallel to the plane of the celestial meridian; therefore the *terrestrial meridian* may be defined to be the curve in which the vertical lines at the different points are parallel to the plane of the same celestial meridian, which will be a plane curve only when the Earth is a solid of revolution. The terrestrial meridian, determined by trigonometrical measurement, is not exactly the same as the above, as will appear from the mode of describing it. To measure an arc of the meridian trigonometrically, a horizontal triangle ABC (fig. 6.) is formed, having one angle coinciding with A, the beginning of the arc to be measured $Aabc$, and the other angles with known objects D and C: having determined the length of the first portion Aa, and its direction with respect to the sides of the triangle ABC; to find a second portion ab, a new horizontal triangle BDC is formed, and ab is drawn in the plane BDC, so as to make the same angle with BC that Aa does; in the same manner, the parts ab, bc make equal angles with BD, and so on for the remaining parts of the whole arc to

be measured. The triangle *ABC* is not in the same plane with *BDC*; therefore, *Aa* produced must fall above *ab*, and in the same manner the other small parts of the line *Abc*, which may be considered tangents to the Earth's surface at their middle points, will, when produced, fall above the adjacent portions. The line thus described is called a Geodesic line, and possesses curious properties[*]; it only coincides exactly with the terrestrial meridian, when the Earth is a solid of revolution; the difference of lengths of these two lines, is, however, so small that it may be safely neglected in any calculations.

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[*] The curvature of a geodesic line at any point lies in a plane perpendicular to the horizon at that point. Let *Aa*, *ab* (fig. 7.) be two consecutive arcs, so small that they may be considered straight lines; produce *Aa* to *a*, and let the sphere described round *a*, cut the consecutive horizontal planes *Caa*, *Cab*, in the arcs *Ca*, *Cb*, and the plane *Aab*, in which the curvature lies in the arc *ba*; then, from the mode of drawing a geodesic line $Cb = Ca$, and the triangle *Cba* is isosceles; therefore

$$\cos Cba = \cot Cb \, . \tan \tfrac{1}{2} \, . \, bd$$

$$= 0, \text{ since } baa \text{ is evanescent;}$$

therefore, *Cba* is a right angle, and the plane *Aab*, in which the curvature of the arc *Aab* lies, is perpendicular to the horizontal plane *Cab*.

The geodesic line is also the shortest line that can be drawn on the surface between any two points of its course. For let *Aa*, *ab* (fig. 8.) be two consecutive small arcs of the shortest line, considered as a polygon of an infinite number of sides; and let the horizontal planes in which they lie intersect in the line *CB*. Then, since *Aab* is part of the shortest line that can be drawn on the surface, $Aa + ab$ must be less than the sum of any two other straight lines drawn from *A* and *b* to the line *CB*.

Draw *Am*, *bn*, perpendicular to *CB*, and let $Am = m$, $bn = n$, $mn = c$, $ma = x$;

$$\therefore \sqrt{x^2+m^2} + \sqrt{\overline{c-x}|^2 + n^2} \text{ must be a minimum;}$$

$$\therefore \frac{x}{\sqrt{x^2+m^2}} - \frac{c-x}{\sqrt{\overline{c-x}|^2+n^2}} = 0;$$

∴

35. *To find the compression by means of a seconds' pendulum, considering the Earth as a spheroid of equilibrium revolving round its minor axis.*

The length of a pendulum oscillating seconds in any latitude varies as the force of gravity at the same place, which, upon the hypothesis of the Earth's being a spheroid of equilibrium, varies as the normal [*]. If, therefore, p, p' be the lengths of two pendulums oscillating seconds in latitudes l, l', and c be the compression, the equatoreal radius being unity.

$$p : p' :: \frac{\overline{1-e^2}}{\sqrt{1-e^2\sin^2 l}} : \frac{\overline{1-e^2}}{\sqrt{1-e^2\sin^2 l'}}$$

$$:: \sqrt{1-e^2\sin^2 l'} : \sqrt{1-e^2\sin^2 l}$$

$$:: 1-\tfrac{1}{2}e^2\sin^2 l' : 1-\tfrac{1}{2}e^2\sin^2 l$$

$$:: 1-c\sin^2 l' : 1-c\sin^2 l;$$

$$\therefore\ p-pc\sin^2 l = p'-p'c\sin^2 l';$$

$$\therefore\ c = \frac{p-p'}{p\sin^2 l - p'\sin^2 l'}.$$

The mean compression deduced from the lengths of seconds' pendulums $= \frac{1}{310}$, which nearly coincides with that from the measurement of degrees.

36. Besides the above modes of determining the com-

$$\therefore\ \frac{ma}{Aa} = \frac{na}{ba};$$

$$\therefore\ \cos Aam = \cos ban;$$

$$\therefore\ Aam = ban,$$

which is the property of the geodesic line.

[*] See Robison's *Philosophy*, Vol. I. p. 463.

pression of the Earth, there is a third method by observations on the Moon. The Moon's motion is subject to several inequalities, which can be determined by observation, or calculation; two of these, which are observed to be 6″ and — 8″, are found by calculation to depend on the compression, and the comparison of the calculated and observed inequalities, give, from the first, a compression $= \frac{1}{305}$; and from the second $= \frac{1}{304.6}$, which nearly coincide with each other, and with the mean compression deduced from the measurement of degrees. If the compression were $\frac{1}{230}$, as found by the comparison of degrees in France, the above inequalities, instead of being 6″.8 and — 8″, would be 11″.5 and — 13″.5.

On the subjects of this chapter the reader may consult Playfair, *Edinburgh Transactions,* Vol. V. Puissant, Geodesie. Delambre, *Arc du Meridien.*

CHAP. III.

TO DETERMINE THE MERIDIAN LINE OF ANY PLACE.

37. THE principal observations of the heavenly bodies are made as they cross the plane of the meridian, which renders the exact determination of this plane of great importance. The meridian plane may be found, when its intersection with the horizon or the meridian line is determined, by suspending two plumb-lines over different points of the meridian line, since both of them, being perpendicular to the horizon, are in the plane of the meridian.

If a telescope moveable round a fixed horizontal axis be so placed, that one of the plumb-lines seen through it hides the other, the plane in which the telescope moves, is that of the meridian, and a telescope so adjusted, is called a *Transit Instrument,* because the heavenly bodies are seen through it at the instant of their transits or passages over the meridian.

38. *If the declination of a heavenly body be invariable, the azimuths corresponding to equal altitudes, on each side of the meridian, are equal.*

Let Z, (fig. 9.) be the zenith of the observer, S and S' two positions of a star at equal altitudes, and therefore at equal zenith distances ZS and ZS', P the pole; then PZ is the co-latitude, and PS, PS' are the co-declinations of S and S', which are equal, since the declination of S is supposed to be invariable; therefore the three sides of the triangle PZS are equal to the sides of PZS', and the expressions for the cosines of the angles PZS, PZS', must be the same; therefore these angles are equal, or PZS the azimuth of $S = PZS'$ that of S'.

39. *To draw a meridian line by the shadow of a vertical gnomon on a horizontal plane.*

The shadow of a vertical rod lies in the vertical plane passing through the Sun and rod; therefore the angle which any shadow makes with the meridian line, is equal to the Sun's azimuth; and since the length of the shadow : the length of the rod :: 1 : tan Sun's altitude; therefore at equal altitudes of the Sun before and after noon, the shadows of the rod are equal. If, then, with the foot of the rod as center, and the length of any shadow before noon as radius, a circle be described, and the point be observed where the Sun's shadow after noon meets the circumference of the same circle, the Sun's altitudes, and therefore the Sun's azimuths at these two instants are equal, and the line bisecting the angle between the two shadows, is the meridian.

The exact vertical position of the gnomon must be ascertained by a plumb-line, and the horizontality of the plane on which the shadow moves, by a level.

40. The objections to this mode of drawing a meridian line are, the difficulty of noting exactly the extremity of the shadow, and that the Sun's declination is not invariable, as supposed in the proof. The error arising from the first cause, has been attempted to be removed, by making the extremity of the gnomon globular, and taking the center of its shadow for the extremity of the rod's shadow, or by placing a thin lamina of metal, pierced with a hole at its center, at the end of the rod, and observing where the spot of light falls, or instead of the lamina of metal a convex lens, which collects the rays from the Sun into its focus. All these methods diminish the error arising from the ill-defined shadow of the extremity, and it may be more lessened by taking the mean of several meridian lines determined in this way for the true meridian.

41. *To correct the place of the meridian, found as above, for the Sun's change of declination.*

Let *Z*, (fig. 10.) be the zenith, *P* the pole, *S* the Sun before noon. Then PZ $(90-l)$ is the co-latitude, l being the altitude of the pole, or latitude of the place, PS $(90^0-\delta)$ the co-declination, and ZS the zenith distance of the Sun. Let $A=PZS$ be the angle between the shadow and meridian line, or the Sun's azimuth, and A', δ' the values of A and δ at the second observation after noon.

$$\text{Then } \cos A = \frac{\cos PS - \cos PZ \,.\, \cos ZS}{\sin PZ \,.\, \sin ZS}$$

$$= \frac{\sin\delta - \sin l \,.\, \cos z}{\cos l \,.\, \sin z},$$

$$\cos A' = \frac{\sin\delta' - \sin l \,.\, \cos z}{\cos l \,.\, \sin z};$$

$$\therefore \cos A' - \cos A = \frac{\sin\delta' - \sin\delta}{\cos l \,.\, \sin z},$$

$$\text{or, } \sin\tfrac{1}{2}\,.\,(A+A')\,.\,\sin\tfrac{1}{2}\,.\,(A-A') = \frac{\sin\tfrac{1}{2}.(\delta'-\delta).\cos\tfrac{1}{2}(\delta'+\delta)}{\cos l \,.\, \sin z}.$$

Or, since the angles $\delta'-\delta$ and $A-A'$ are very small,

$$\tfrac{1}{2}.(A-A')\,.\,\sin\tfrac{1}{2}\,.\,(A'+A) = \frac{\tfrac{1}{2}\,.\,(\delta'-\delta)\,.\,\cos\tfrac{1}{2}(\delta'+\delta)}{\cos l \,.\, \sin z}.$$

Let t be the time by the clock between the two positions of the shadow, Δ the declination of the Sun at noon, Δ' at the preceding noon, $24+x$ the number of hours between the two noons. Then the change of declination may be considered uniform, during one day; therefore

$$24+x : t :: \Delta - \Delta' : \delta' - \delta \text{ nearly,}$$

$$\text{and } \delta' + \delta = 2\Delta \text{ nearly;}$$

$$\therefore \tfrac{1}{2}.(A-A') = \tfrac{1}{2}\,.\,\frac{t\,.\,(\Delta-\Delta')}{24+x}\,.\,\frac{\cos\Delta}{\cos l \,.\, \sin z \,.\, \sin\tfrac{1}{2}(A'-A)}.$$

In this equation z is known from the expression, length of shadow $=$ rod $\tan z$, and l and δ must be supposed given, at least approximately, for a small error in these quantities will not have much influence on the result, if $\Delta - \Delta'$ be exactly known.

The angle $\frac{1}{2}.(A' - A)$ thus determined, is the angle between the true meridian and the approximate meridian, found by bisecting the angle $A' + A$ between the two shadows.

42. There is no need of applying the correction for the change of declination when the Sun is near the solstices, for then, as will be proved hereafter, the Sun's declination may be considered invariable during the same day.

43. *When the vertical plane in which a transit instrument moves, nearly coincides with the meridian, to find the deviation.*

Let Z (fig. 11.) be the zenith, P the pole, S the place of a star in its parallel Sa, a the place where it crosses the meridian, ZS the vertical plane in which the telescope moves, then ZP $(90^0 - l) =$ the co-latitude, and PS $(90 - \delta) =$ the co-declination of the star.

$$\text{And } \frac{\sin SPZ}{\sin SZP} = \frac{\sin ZS}{\sin PS},$$

or, since the angle SZP, SPZ are very small, and ZS nearly equal ZA,

$$\frac{SPZ}{SZP} = \frac{\sin(\delta - l)}{\cos\delta};$$

$$\therefore SPZ = SZP.(\cos l.\tan\delta - \sin l).$$

Let T be the sidereal time of the star's transit over the meridian, t the time of the observation at S, both expressed in degrees at the rate of 15^0 to an hour.

$$\text{Then } T = t + SZP.(\cos l.\tan\delta - \sin l).$$

Similarly, if T', t', δ' are values of T, t, δ for another star,

$$T' = t' + SZP.(\cos l.\tan\delta' - \sin l);$$

$$\therefore\ T-T'=t-t'+SZP\,.\cos l\,.(\tan\delta-\tan\delta')$$

$$=t-t'+SZP\,.\cos l\,.\frac{\sin(\delta-\delta)'}{\cos\delta\,.\cos\delta'};$$

$$\therefore\ SZP=\{T-T'-(t-t')\}\,.\frac{\cos\delta\,.\cos\delta'}{\cos l\,.\sin(\delta-\delta')}\,.$$

In which equation $T-T'$ the difference of the right ascensions of the two stars, is known from the catalogues in which all stars are registered with their right ascensions and declinations, and $t-t'$ may be observed; therefore PZS, the deviation of the transit instrument from the meridian, may be found and corrected.

In order that the observed error $T-T'-(t-t')$, caused by the deviation of the transit instrument, may be the greatest possible, those stars ought to be chosen, in which $\delta-\delta'=90^0$, nearly; so that if the one star cross the meridian near the zenith, the other ought to cross near the horizon, or if the declination of the one star is small, that of the other ought to be near 90^0.

44. *To find the same from the observed superior and inferior transits of the same circumpolar star.*

Constructing as before, let $SS'a'a$ be the whole parallel described by the star, S' and a' the places of S and a at the inferior transit. Then

$$\frac{\sin S'Pa'}{\sin PZS'}=\frac{\sin S'PZ}{\sin S'ZP}$$

$$=\frac{\sin S'Z}{\sin PS'}\,.$$

But since the angles $S'Pa'$, PZS' are very small, they may be taken for their sines, and

$$\sin S'Z=\sin za'\ \text{nearly}=\sin(90-l+90-\delta)=\sin(l+\delta);$$

$$\therefore\ \frac{SPa'}{PZS'}=\frac{\sin(l+\delta)}{\cos\delta}$$

$$=\sin l+\tan\delta\,.\cos l;$$

$$\therefore\ S'Pa' = SZP\,.(\cos l\,.\tan\delta + \sin l),$$

$$\therefore\ T' = t' - SZP\,.(\cos l\,.\tan\delta + \sin l);$$

$$\text{but } T = t + SZP\,.(\cos l \tan\delta - \sin l)$$

$$\therefore\ T - T' = t - t' + 2SZP\,.\cos l\,.\tan\delta.$$

Now $T - T' = 12^{\text{h}}$ or 180^0;

$$\therefore\ SZP = \tfrac{1}{2}\,.\,\frac{180^0 - (t - t')}{\cos l\,.\tan\delta}\,.$$

This result depends only on the latitude of the place, the observed interval of the superior and inferior transits, and the star's declination, and does not require the difference of right ascension as the former does; but as the transits take place after an interval of twelve hours, one of them for the greater part of the year must be observed by day-light, and the stars near the equator, the time of whose transits can be observed most accurately from the rapidity of their motions, cannot be used.

45. In dialling, where the plane of the meridian is required, it is usual to find it by suspending a circle of metal or wood by a string, in such a manner, that its own weight may keep it vertical, and observing when two stars, whose right ascensions are nearly equal, or differ by 180^0, are seen in the plane of the circle. At this instant the plane nearly coincides with the meridian, and would accurately, if the difference of right ascension of the stars were exactly 0^0 or 180^0.

The pole star, and ϵ in the tail of the great bear, whose difference of right ascension $= 177^0.\ 43'.\ 11''$; or α Ophinci, and β Draconis, where the difference of right ascension is $7''$, are adapted for finding the meridian by this method*.

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* For a mode of correcting the meridian so found, see Brinkley's *Astronomy*, p. 161.

CHAP. IV.

TO FIND THE LATITUDE OF A PLACE ON THE EARTH'S SURFACE.

The latitude of a place on the Earth's surface may be found in various ways, and the mode to be adopted in any particular case must be determined by the observations which it is possible to make.

46. *To find the latitude by means of a transit instrument*[*].

1. If the Sun's declination is known, observe the zenith distance of the Sun; then

the latitude = the declination + the Sun's meridian zenith distance.

2. The latitude = the declination ± the observed meridian zenith distance of a known star; + if the star crosses the meridian between the zenith and equator, and − if the star crosses the meridian between the zenith and pole.

3. If two altitudes A and a, of the same circumpolar star, be observed at the times of its superior and inferior transit over the meridian,

the latitude = the altitude of the pole,
= A − the star's north polar distance.

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[*] By a transit *here* is understood any instrument, such as a mural circle or quadrant, with which altitudes or zenith distances may be observed in the plane of the meridian; this is not the *proper* use of a transit instrument, which is only intended for observing the time of a star's transit.

Similarly, the latitude $= a +$ the star's north polar distance;

$\therefore$ the latititude $= \frac{1}{2}.(A+a)$, by addition.

47. These methods may be used with any instrument adapted to observing angles in the plane of the meridian. Or, if the exact time of the Sun or star's being on the meridian is known, they may be used with an instrument for taking altitudes, not fixed in the plane of the meridian, in which latter case they are applicable to finding the latitude at sea.

48. *To find the latitude from two equal altitudes of the Sun, observed before and after noon.*

Let Z (fig. 9.) be the zenith, P the pole, S and S' the two positions of the Sun at the equal zenith distances ZS, ZS'; then, if the Sun's declination, which changes very slowly, (about 23^0 in three months) be considered the same at the two observations, $PS = PS'$ and the angle

$$SPZ = S'PZ = \tfrac{1}{2}.(SPS').$$

Now if t be the hours by the clock between the observations, and T the hours between the preceding and succeeding noons, considering the Sun's motion in right ascension be uniform during that interval,

$$T : t :: 360^0 : SPS' (2\alpha),$$

which determines SPZ (α), and from this angle, the Sun's declination $(90 - PS)$, and the observed altitude $(90^0 - ZS)$, PZ the co-latitude may be found.

Calculation. Let $PZ = 90 - l$, $PS = 90 - \delta$, $ZS = 90 - a$,

$$\text{then } \sin a = \cos l.\cos\delta.\cos\alpha + \sin l.\sin\delta.$$

Let ϕ be a subsidiary angle, determined by the equation

$$\tan\phi = \cos\alpha.\cot\delta;$$

$$\therefore \sin a = \cos l.\sin\delta.\tan\phi + \sin l.\sin\delta$$

$$= \sin\delta.\frac{\cos l.\sin\phi + \sin l.\cos\phi}{\cos\phi}$$

$$= \frac{\sin \delta . \sin (l+\phi)}{\cos \phi};$$

$$\therefore \sin (l+\phi) = \frac{\sin a . \cos \phi}{\sin \delta},$$

which determines $l+\phi$, and, therefore, the latitude l.

Cor. 1. In the same manner, the latitude may be found by two equal altitudes of the same star.

Cor. 2. A mode of correcting the error in the determination of the angle ZPS, introduced by supposing the Sun's declination invariable, is given in the Chapter on Time.

49. **Cor. 3.** The only use of the two altitudes is to determine the angle SPZ, but if the time between the first observation and the instant of the Sun's passage over the meridian be known, that angle is found at once by the proportion

$$T : t :: 360^0 : SPZ,$$

and one altitude will then suffice for determining the latitude.

As the Sun may be obscured by clouds at the instant when the second observation ought to be made, the following method of determining the latitude of *any* two observed altitudes is frequently useful.

50. *To find the latitude and hour angle, from two altitudes of the Sun, and the time between.*

Let Z (fig. 12.) be the zenith, P the pole, S and S' the two positions of the Sun; then, from the time between the observations, SPS' (a) may be found as in Art. 48.

Let $ZS = z$, $ZS' = z'$; which are the complements of the observed altitudes,

$PS = \Delta$, $PS' = \Delta'$, the co-declinations of S and S',

$SS' = \theta$, and $l = 90 - PZ$, the latitude of Z.

Then, in the triangle PSS';

$$\tan \tfrac{1}{2} . (PSS' + PS'S) = \frac{\cos \frac{1}{2} (\Delta' - \Delta)}{\cos \frac{1}{2} (\Delta' + \Delta)} . \cot \frac{a}{2},$$

$$\tan \tfrac{1}{2} . (PSS' - PS'S) = \frac{\sin \frac{1}{2} . (\Delta' - \Delta)}{\sin \frac{1}{2} . (\Delta' + \Delta)} . \cot \frac{a}{2},$$

$$PSS' = \tfrac{1}{2} . (PSS' + PS'S) + \tfrac{1}{2} . (PSS' - PS'S),$$

$$\sin \theta = \frac{\sin \Delta' . \sin a}{\sin PSS'} .$$

In the triangle ZSS',

$$\sin \tfrac{1}{2} (ZSS') = \sqrt{\frac{\sin \frac{1}{2} . (z' + z - \theta) . \sin \frac{1}{2} . (z' + \theta - z)}{\sin \theta . \sin z}},$$

$$PSZ = PSS' - ZSS'.$$

In the triangle ZPS,

$$\sin l = \cos z . \cos \Delta + \sin z . \sin \Delta . \cos PSZ.$$

$$\sin ZPS = \sin ZSP . \frac{\sin z}{\cos l} .$$

The last equation but one may be replaced by the formulæ

$$\tan \phi = \cos PSZ . \tan \Delta,$$

$$\sin l = \frac{\cos (z - \phi) . \cos \Delta}{\cos \phi},$$

which are adapted to logarithmic calculation.

These equations determine successively PSS', θ, ZSS', ϕ, and, lastly, l, and ZPS, which are the quantities required.

The figure represents the two positions of the Sun on the same side of the meridian, but the proof is the same when they are on different sides.

51. The Sun's declination, which is required in the above method, is subject to daily alteration, and is registered in tables which contain the declinations corresponding to the noon of every day under a given meridian. From these registered declinations, the declination for any intermediate time is found, by taking a part of the whole increment of declination proportional to the time from the preceding noon; thus if δ and δ' be two declinations from the tables for the times of the preceding and succeeding noon under the given meridian, and T^h the time between the two noons, then $\delta' - \delta$ is the increment of declination in T^h, therefore, in t^h, from the time when the declination was δ, the increment of declination is $\frac{t}{T} \cdot (\delta' - \delta)$, and the whole declination $= \delta + \frac{t}{T} \cdot (\delta' - \delta)$. If, therefore, at the place where the observations are made we have a chronometer which indicates the time at the place for which the tables are calculated, the declinations, and, therefore, the co-declinations Δ and Δ', may be found. This remark will be better understood after the perusal of the Chapter on Time.

52. Cor. 1. In the same manner, the latitude may be found from the observed altitudes of two known stars, and the time between. If the observations are cotemporary, a = the difference of the star's right ascensions (d); if they are separated by an interval of t sidereal hours, $a = d \pm 15^0 t$, + when the star on which the second observation is made, is west, and $\sim$ when it is east of the other.

53. Cor. 2. If the Sun's declination be considered invariable between the observations, or, if the latitude be deduced from two altitudes of the same star, the four first equations of Art. 50, reduce themselves to

$$\tan PSS' = \frac{\cot \frac{1}{2} a}{\cos \Delta},$$

$$\sin \tfrac{1}{2}\theta = \sin \tfrac{1}{2}a \,.\, \sin \Delta\text{*},$$

The remainder of the solution is the same as before.

In this manner the general problem is often solved; the north polar distance is supposed to remain equal to $\frac{1}{2}(\Delta + \Delta')$ between the observations, and the error of from 1′ to 5′, thus introduced, is got rid of by a subsequent correction.

54. When the latitude is nearly known, a more correct value may often be obtained from two altitudes of the Sun, and the time between, by the following method, which is called, from its inventor, *Douwe's Method*, and is much used at sea.

With the same assumptions as before, let l' be the approximate latitude, h and h', the hour angles, and Δ the Sun's north polar distance, which is here taken equal to half the sum of the polar distances at the two observations, and is supposed to remain constant.

$$\text{Then } \cos z = \cos l \sin \Delta \cos h + \sin l \cos \Delta \ldots . (1)$$

$$= \cos l' \sin \Delta \cos h + \sin l' \cos \Delta \text{ nearly,}$$

$$\text{and } \cos z' = \cos l' \sin \Delta \cos h' + \sin l' \cos \Delta ;$$

$$\therefore \cos z - \cos z' = \cos l' \sin \Delta \,.\, (\cos h - \cos h') ;$$

$$\therefore \sin \tfrac{1}{2} \,.\, (z' - z) \,.\, \sin \tfrac{1}{2} (z' + z)$$

$$= \cos l' . \sin \Delta . \sin \tfrac{1}{2} \,.\, (h' - h) \,.\, \sin \tfrac{1}{2} \,.\, (h' + h).$$

In this equation, $h' - h$ is known from the observed interval, when the observations are made on the same side of the meridian, and $h' + h$, when they are made on different sides, and either of these being known, the other may be deduced from it by the equation, and thus h may be found.

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* These equations may be obtained at once from Napier's rules, if the isosceles triangle *PSS′* be divided into two equal right-angled triangles.

But from equation (1),

$$\cos z = \sin (l + \Delta) - 2 \cos l \sin \Delta \sin^2 \tfrac{1}{2} h;$$

$$\therefore \sin (l + \Delta) = \cos z + 2 \cos l' \sin \Delta \sin^2 \tfrac{1}{2} h \text{ nearly,}$$

from which $l + \Delta$, and, therefore, l may be found.

If the subsidiary angle ϕ be determined from the equation

$$\cot \phi = \frac{2 \cos l' \sin \Delta}{\sin z} . \sin^2 \tfrac{1}{2} h,$$

$$\sin (l + \Delta) = \frac{\sin (z + \phi)}{\sin \phi},$$

which is adapted to logarithmic computation.

If the value of l thus deduced differs much from the approximate value l', the operation must be repeated with l instead of l', when the method becomes longer than the direct method, without being so certain. See Delambre, Chap. xxxvi.

55. When the latitude is found *at sea*, from two altitudes of the Sun and the time between, a correction must be applied for the ship's change of place between the observations.

Let P (fig. 13.) be the pole, Z the zenith, and S the Sun at the first observation, Z' and S' at the second. Let $SZZ' = m$, the arc $ZZ' = n$; then, m is known from the direction of the ship's motion, and if s be the actual space moved over on the Earth's spherical surface, which is known from the ship's rate and the time, $n = \frac{s}{r}$, r being the Earth's radius. Let a, a' be the observed altitudes of S and S', and $a_1 = 90 - Z'S$ the altitude which would have been observed, if the zenith at the first observation had been Z'. Let $a_1 = a + y$, then, from the triangle $Z'ZS$,

$$\sin (a + y) = \sin n . \cos a . \cos m + \sin a . \cos n.$$

Differentiating this equation with respect to the very small quantities y and n, and denoting by y', y'', &c., the differential coefficients of y, and by y_0', y_0'', &c., the same quantities when n vanishes, we obtain

$$\cos(a+y)\,.\,y' = \cos a\,.\,\cos m\,.\,\cos n - \sin a\,.\,\sin n$$

$$\cos(a+y)\,y'' - \sin(a+y)\cdot y'^2 = -\cos a\,.\,\cos m\,.\,\sin n - \sin a\cos n;$$

$$\therefore\ y_0 = 0,$$

$$\cos a y_0' = \cos a\,.\,\cos m,$$

$$\cos a\,.\,y'' - \sin a\,.\,y_0'^2 = -\sin a;$$

$$\therefore\ y_0' = \cos m,\ y_0'' = -\tan a\,.\,\sin^2 m;$$

therefore, by Maclaurin's theorem,

$$y = \cos m\,.\,n - \tfrac{1}{2}\,.\,\tan a\,.\,\sin^2 m\,.\,n^2,$$

neglecting powers of n above the second.

$$\text{But } a_1 = a + y \text{ and } n = \frac{s}{r};$$

$$\therefore\ a_1 = a + \cos m\,.\,\frac{s}{r} - \tfrac{1}{2}\,.\,\tan a\,.\,\sin^2 m\,.\,\frac{s^2}{r^2}$$

$$= a + \cos m\,.\,\frac{s}{r\sin 1''} - \tfrac{1}{2}\,.\,\tan a\sin^2 m\,.\,\frac{s^2}{r^2\sin 1''},$$

when a_1 and a are expressed in seconds*.

Having thus found a_1, we may proceed, as before, to find the latitude of the ship, at the second observation, from the altitudes a_1, a', and the time between.

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* If a'' be the seconds in an arc, the length of which, expressed in parts of the radius, is a, $a = a'' \times$ length of the arc of $1''$;

but the arc of $1'' = \sin 1''$ nearly;

$$\therefore\ a = a''.\sin 1'', \text{ and } a'' = \frac{a}{\sin 1''}.$$

56. Cor. If the spheroidal figure of the Earth be taken into consideration, r must be taken for the normal at either place of observation.

57. *To determine the latitude from zenith distances of a heavenly body, observed very near the meridian.*

Let $z = PS$ (fig. 10.) be the zenith distance of the Sun or of a known star S, observed very near the meridian, $l = 90^0 - PZ$, the latitude required, $\delta = 90^0 - PS$ the star's known declination, and $h = ZPS$ the hour angle.

$$\text{Then } \cos z = \cos l \,.\, \cos \delta \,.\, \cos h + \sin l \sin \delta$$

$$= \cos (l - \delta) - 2 \cos l \,.\, \cos \delta \,.\, \sin^2 \tfrac{1}{2} h.$$

Let $z = m + y$, m being the star's meridian zenith distance, and y a very small quantity to be determined; then $m = l - \delta$, if the star be between the zenith and equator, and $\delta - l$, if between the zenith and pole. Let, also,

$$2 \cos l \cos \delta \,.\, \sin^2 \tfrac{1}{2} h = x,$$

and denote, as before, the differential coefficients of y with respect to x, by y', y'';

$$\therefore \cos (m + y) = \cos (l - \delta) - x,$$

$$\sin (m + y) y' = 1,$$

$$\sin (m + y) y'' + \cos (m + y) y'^2 = 0.$$

Making $x = 0$ in these equations, and denoting by y_0', y_0'', the corresponding values of y,

$$y_0 = 0,$$

$$\sin m \,.\, y_0' = 1,$$

$$\sin m \,.\, y_0'' + \cos m \,.\, y_0'^2 = 0;$$

$$\therefore y_0' = \frac{1}{\sin m}, \text{ and } y_0'' = -\cot m \,.\, \frac{1}{\sin^2 m};$$

therefore, by Maclaurin's theorem, replacing x and m by their values,

$$y = \frac{2 \cos l . \cos \delta . \sin^2 \frac{1}{2} h}{\sin (l - \delta)}$$

$$- \tfrac{1}{2} . \cot (l - \delta) . \left(\frac{2 \cos l . \cos \delta \sin^2 \frac{1}{2} h}{\sin (l - \delta)} \right)^2 + \&c.$$

Or, expressed in seconds, (Note to Art. 55.)

$$y = \frac{2 \cos l . \cos \delta . \sin^2 \frac{1}{2} h}{\sin 1'' . \sin (l - \delta)}$$

$$- \tfrac{1}{2} . \cot (l - \delta) . \sin 1'' . \left(\frac{2 \cos l . \cos \delta . \sin^2 \frac{1}{2} h}{\sin 1'' . \sin (l - \delta)} \right)^2 + \&c.$$

In calculating the coefficient of $\sin^2 \frac{1}{2} h$, l may be taken equal the least observed zenith distance $+\delta$. To find h itself; if a transit instrument be used, the clock will shew the interval between the time of observation and the transit, from whence h may be found, as in Art. 49. If there be no transit instrument, an equal zenith distance must be observed on the other side of the meridian, from which h may be calculated, as in Art. 48.

Having by these means got the correction y, $z - y$, the meridian zenith distance is found. If, therefore, several meridian zenith distances of the same star be taken near the meridian, and we call Z the mean of the values of z, and Y of y.

The true meridian zenith distance $= Z - Y$,

and the true latitude $= \delta \pm (Z - Y)$.

In general the first term of the value of y is sufficient, and the second, which is easily deduced from it, need not be resorted to, unless the star be at a considerable distance from the meridian.

The pole star, which is a star of the first magnitude very near the north pole, on account of its slow motion, may be employed with greater advantage than any other in determining the latitude by this method; with this star, y is very small for more than a quarter of an hour before and after the star's transit.

This method, by which the latitude is found with very great accuracy, may be used by land or sea; in the latter case h must be determined as in Art. 48.

58. Cor. The change of altitude, or zenith distance near the meridian, varies as $\sin^2 h$, and therefore as h^2 nearly, or as the square of the interval between the time of observation, and the time of transit.

59. If H be the meridian altitude, h and h' two other altitudes observed near the meridian, T, t, t', the corresponding times; then, because the change of altitude, near the meridian, varies as the square of the interval between the time of observation and the time of transit,

$$H-h : H-h' :: (T-t)^2 : (T-t')^2;$$

$$\therefore H.(T-t')^2 - h.(T-t')^2 = H.(T-t)^2 - h'.(T-t)^2;$$

$$\therefore H = \frac{h.(T-t')^2 - h'.(T-t)^2}{(T-t')^2 + (T-t)^2}.$$

60. Let h, $h+a$, $h+a'$ be three altitudes of the Sun or of a star, observed very near the meridian,

T, $T+b$, $T+b'$. . . . corresponding times by the clock,

$T+t$ the time corresponding to the meridian altitude H.

Then, $H-h : H-h-a :: t^2 : (t-b)^2$;

$$\therefore (H-h).(t^2-2tb+b^2) = (H-h).t^2 - at^2;$$

$$\therefore at^2 = (H-h).(2tb-b^2).$$

Similarly, $a't^2 = (H-h) \, . \, (2tb' - b'^2)$;

$$\therefore \frac{a}{a'} = \frac{2tb - b^2}{2tb' - b'^2};$$

$$\therefore t = \tfrac{1}{2} \, . \frac{ab'^2 - a'b^2}{ab' - ba'} \ldots\ldots\ldots\ldots (1).$$

$$\text{But } H - h = \frac{at^2}{2tb - b^2};$$

therefore, substituting the preceding value of t,

$$H = h + \frac{(ab'^2 - b^2a')^2}{4 \, . \, bb', (b' - b) \, . \, (ab' - ba')} \ldots (2).$$

The equation (2) determines the meridian altitude H, from which the latitude $\delta \pm (90 - H)$ may be found. From equation (1) we obtain $T + t$, the time by the clock of the star's transit over the meridian.

This method, as Littrow remarks, has the advantage of requiring neither the approximate value of the latitude, the hour angle, nor the star's declination, and since it contains only the differences of times for a small interval, may be used with a watch which gains or loses several minutes in the day. It requires, however, that the differences of times and altitudes be observed very exactly.

61. The pole star, on account of the smallness of its north polar distance ($1^\circ. 38'. 47''$) may be advantageously used for finding the latitude in any point of its circular course.

Let a, an altitude corresponding to the hour angle h, $= l + y$, where y is always very small, and in no case greater than $\pm 1^\circ. 38'. 47''$ (Δ); then

$$\sin(l + y) = \cos l \sin \Delta \, . \cos h + \sin l \, . \cos \Delta;$$

therefore, as in Art. 55.

$$y = \cos h \, . \, \Delta - \tfrac{1}{2} \sin^2 h \, . \tan l \, . \, \Delta^2, \text{ nearly};$$

therefore the latitude $= a - y$

$$= a - \cos h \Delta + \tfrac{1}{2} \sin^2 h \, . \tan l \, . \, \Delta^2.$$

For a first value, l may be taken $= a - \cos h\Delta$, from which the coefficient of Δ^2 may be calculated; therefore determining n latitudes l, l', &c., in this manner, the true latitude

$$= \frac{1}{n} . (l + l' + \&c.).$$

The observations required in Articles 57 and 61 may be made with any instrument for taking altitudes or zenith distances, but the formulæ are principally intended to be applied to observations made with the repeating circle, which is a portable instrument well adapted to observing a number of angles in succession. The meridian altitude of the Sun or a star thus obtained from several altitudes taken with this instrument very near the meridian, is nearly as correct as that found by the more costly instruments which are fixed in the plane of the meridian. For a description of the repeating circle, see *Biot Astronomie*, Chap. xx. p. 273.

62. *Given the latitude of a place on the Earth's surface considered as an oblate spheroid, to find the latitude measured from the center.*

Let $l = PGe$, (fig. 5.) be the latitude of P, $l' = PCe$ the latitude from the center of the Earth C, Ce the equatoreal radius $= a$, and Cp the polar radius $= b$.

$$\text{Then, } \tan l' : \tan l :: GN : CN :: b^2 : a^2;$$

$$\therefore \tan l' = \frac{b^2}{a^2} . \tan l = n . \tan l, \left(\text{making } \frac{b^2}{a^2} = n\right);$$

$$\therefore \frac{e^{2l'\sqrt{-1}} - 1}{e^{2l'\sqrt{-1}} + 1} = n . \frac{e^{2l\sqrt{-1}} - 1}{e^{2l\sqrt{-1}} + 1};$$

$$\therefore e^{2l'\sqrt{-1}} . (e^{2l\sqrt{-1}} + 1) - e^{2l\sqrt{-1}} - 1$$

$$= e^{2l'\sqrt{-1}} . (ne^{2l\sqrt{-1}} - n) + ne^{2l\sqrt{-1}} - n;$$

$$\therefore e^{2l'\sqrt{-1}}.\left\{1+n+\overline{1-n}.e^{2l\sqrt{-1}}\right\}$$

$$=1-n+\overline{1+n}.e^{2l\sqrt{-1}};$$

$$\therefore e^{2l'\sqrt{-1}}.\left\{1+me^{2l\sqrt{-1}}\right\}=m+e^{2l\sqrt{-1}},\text{ if } m=\frac{1-n}{1+n};$$

$$\therefore e^{2l'\sqrt{-1}}=e^{2l\sqrt{-1}}.\frac{1+me^{-2l\sqrt{-1}}}{1+me^{2l\sqrt{-1}}};$$

$$\therefore 2l'\sqrt{-1}=2l\sqrt{-1}+\log\left(1+me^{-2l\sqrt{-1}}\right)$$

$$-\log\left(1+me^{2l\sqrt{-1}}\right)$$

$$=2l\sqrt{-1}-m.\left(e^{2l\sqrt{-1}}-e^{-2l\sqrt{-1}}\right)$$

$$+\tfrac{1}{2}m^2.\left(e^{4l\sqrt{-1}}-e^{-4l\sqrt{-1}}\right);$$

$$\therefore l'=l-m.\sin 2l+\tfrac{1}{2}m^2\sin 4l-\tfrac{1}{3}m^3\sin 6l+\&c.$$

$$=l-\frac{1-n}{1+n}.\sin 2l+\tfrac{1}{2}.\left(\frac{1-n}{1+n}\right)^2.\sin 4l-\&c.;$$

$$\therefore\ =l-\frac{a^2-b^2}{a^2+b^2}.\sin 2l+\tfrac{1}{2}.\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.\sin 4l-\&c.$$

63. Cor. 1. Since $\tan l=\frac{1}{n}.\tan l'$, it may be proved, in the same manner, that

$$l=l'-m'.\sin 2l'+\tfrac{1}{2}m'^2\sin 4l'-\&c.,$$

$$\text{where } m'=\frac{1-\frac{1}{n}}{1+\frac{1}{n}}=-\frac{1-n}{1+n};$$

$$\therefore\ l = l' + \frac{1-n}{1+n} \, . \sin 2l' + \tfrac{1}{2} \, . \left(\frac{1-n}{1+n}\right)^2 . \sin 4l' + \&c.$$

64. Cor. 2. If $n = \cos \epsilon$,

$$\frac{1-n}{1+n} = \tan^2 \epsilon\,;$$

$$\therefore\ l' = l - \tan^2 \epsilon \, . \sin 2l + \tfrac{1}{2} \, . \tan^4 \epsilon \, . \sin 4l - \&c.$$

$$l = l' + \tan^2 \epsilon \, . \sin 2l' + \tfrac{1}{2} \, . \tan^4 \epsilon \, . \sin 4l' + \&c.$$

65. Cor. 3. If ϕ be the correction which must be added to the observed latitude, in order to get the latitude reckoned from the Earth's center,

$$\begin{aligned} \phi &= l - l' \\ &= \tan^2 \epsilon \, . \sin 2l - \&c. \\ &= \tan^2 \epsilon \, . \sin 2l' + \&c. \end{aligned}$$

CHAP. V.

ON THE SUN'S MOTION IN THE ECLIPTIC.

66. *To find, by observation, the Sun's motion in right ascension in any time.*

Let t and t', the respective times of transit of the Sun and a fixed star, be observed by the clock; then, if the clock gain or lose uniformly, and $24 + x$ be the hours between two successive transits of the same fixed star,

$24 + x : t' - t :: 360^0$: difference of right ascension of the Sun and star, at the instant the Sun was on the meridian.

If A and a be the differences of right ascension of the Sun and the same fixed star, thus found at different times, $a - A$ is the Sun's motion in the interval, supposing the star's right ascension to be invariable.

67. In finding the Sun's motion in right ascension, it is not necessary that the hand of the clock should go round exactly in 24 sidereal hours, but merely that the clock should gain or lose time uniformly during the same day, and that the whole gain or loss be known.

68. The stars, as will be shewn in the chapters on Aberration and Precession, do not always appear in exactly the same place, but have small apparent changes of position, which can be ascertained; if, therefore, S be the right ascension of the star at the first observation, and $S + e$ at the second,

A = the Sun's right ascension at the first observation $- S$,

a = ... second . . . . . . $-(S+e)$;

$\therefore a - A$ = the Sun's motion in right ascension $- e$;

$\therefore$ the Sun's motion in right ascension $= a - A + e$.

69. *To find the Sun's declination by observation.*

Let the zenith distance of the Sun, when on the meridian, be observed, then, in north latitudes, the observed zenith distance = the latitude of the place − the Sun's declination; therefore the Sun's declination = the latitude of the place − the observed zenith distance.

If the declination, thus found, is negative, the Sun is on the south side of the equator.

In observing the altitude of the Sun, or of any other heavenly body which has apparent magnitude, the altitudes of the upper and lower limb are observed, and the semi-sum of these is taken for the altitude of the center; in the same way to find the time of transit, the times of transit of the western and eastern limb are observed, and their semi-sum taken for the true time of the transit of the center.

70. If the Sun's declination be thus found on every day throughout the year, it is found to be 0 at one time, and afterwards to increase up to $23^{\circ}.\ 28'$, at the end of three months; from this time it decreases, and is again 0, at the end of half a year, when the declination becomes south, and the same changes of south declination recur for the next half year. If the Sun's motion in right ascension (A) in the interval between his declination being 0, and D be observed, it is found that $\sin A \propto \tan D$, which proves that the Sun's motion is in a great circle, which cuts the equator in the points where his declination is 0.

71. *To find the right ascension of the Sun and the obliquity of the ecliptic.*

Let ♈ D, (fig. 14.) be the equator, ♈ S the ecliptic, or plane in which the Sun moves, S and s two observed positions of the

Sun, when on the meridian of the observer, $SD\,(\delta)$, and $sd\,(\delta')$ the corresponding declinations. Let $Dd\,(2a)$ the difference of right ascension of S and s be observed; then if $♈D = x$, and $S♈D$ the angle between the ecliptic and equator, which is called the obliquity, $= \omega$,

$$\sin x = \tan\delta \,.\, \cot\omega,$$

$$\sin(x+2a) = \tan\delta' . \cot\omega;$$

$$\therefore \frac{\sin(x+2a)+\sin x}{\sin(x+2a)-\sin x} = \frac{\tan\delta'+\tan\delta}{\tan\delta'-\tan\delta}.$$

$$\text{Or } \frac{\tan(x+a)}{\tan a} = \frac{\sin(\delta'+\delta)}{\sin(\delta'-\delta)};$$

$$\therefore \tan(x+a) = \frac{\sin(\delta'+\delta)}{\sin(\delta'-\delta)} . \tan a,$$

$$\text{and } \cot\omega = \sin x . \cot\delta.$$

Which equations determine x and ω.

The value of the obliquity thus found $= 23^0.\ 27'.\ 40''$.

72. Cor. 1. Having found the right ascension of the Sun, that of any other heavenly body may be found by adding to it the difference of right ascension of the body and the Sun determined, as in Art. 66.

73. Cor. 2. The right ascension of the Sun, when on the meridian, converted into time at the rate of 15^0 to an hour, will give the sidereal time from the first point of Aries being on the meridian.

74. *Flamstead's method of finding the Sun's right ascension.*

Let $♈DL$ (fig. 14.) be the equator, $♈SL$ the ecliptic, S the Sun *near* the equinox ♈, when on the observer's meridian, D his place referred to the equator by the declination circle SD. Let m and n be places of D similarly determined near the equinox L, where the Sun's declinations δ' and δ'' at succeeding noons, are one greater, and the other less than $SD\,(\delta)$, D' the intermediate place where the declination $=\delta$. Then $Dm\,(2a)$, the difference of the Sun's right ascension at D and m, may be found (66), and $mn\,(2\beta)$ the difference at m and n.

Now the changes of right ascension and declination may be considered uniform during the same day;

$\therefore\ \delta' - \delta'' : \delta' - \delta$ :: the time between the two successive transits of the Sun : time from m to D'

$$:: mn : mD'$$

$$:: 2\beta : mD' = 2\beta \, . \frac{\delta' - \delta}{\delta' - \delta''}.$$

Let A be the Sun's right ascension at the first observation;

$\therefore\ A + 2a$ is the right ascension of m,

and $A + 2a + 2\beta \, . \dfrac{\delta' - \delta}{\delta' - \delta''}$ of D', which equals $180^\circ - A$;

$$\therefore\ A = 90^\circ - a - \beta \, . \frac{\delta' - \delta}{\delta' - \delta''}.$$

The differences $\delta' - \delta$, $\delta' - \delta''$ equal the differences of the observed meridian zenith distances, which renders the knowledge of the latitude not requisite in Flamstead's method, and causes all errors which equally affect the observed zenith distances to vanish. These advantages render the approximate method, in practice more accurate than the preceding, which is theoretically exact.

75. When the Sun is in one of the solstices, the right ascension $=90^0$, and the declination equal the obliquity (ω). If, therefore, we suppose the Sun to be on the meridian of any place, latitude (l) at the time when he is in the solstices, and that the altitudes A' and A' are observed corresponding to the summer and winter solstice.

$$A=90^0-\odot\text{'s zenith dist.}=90-(l-\odot\text{'s dec.})=90-l+\omega.$$

Similarly, $A'=90^0-\odot\text{'s zenith dist.}=90-(l+\odot\text{'s dec.})=90-l-\omega$;

$$\therefore \omega=\tfrac{1}{2}.(A-A'),$$

$$l=90-\tfrac{1}{2}.(A+A').$$

76. As the Sun can only be on one meridian at the time of either solstice, and is never on the same meridian at two successive solstices, the ancients, who made use of this method, employing the greatest and least meridian altitudes found by the shadows of gnomons for the solstitial altitudes, could not have obtained exact results.

77. If a be the length of a vertical gnomon, and s, s', the lengths of the shadows at the time of the solstices, $a=s\tan A$, or $s'\tan A'$; whence, from the equations of Art. 75, we easily get

$$\tan 2\omega=a.\frac{s'-s}{a^2+ss'},$$

$$\tan 2l=a.\frac{s'+s}{a^2-ss'}.$$

78. *To find the relation between the Sun's right ascension, declination, and longitude.*

Let ♈ S (fig. 14.) the Sun's longitude $=l$, ♈ D the right ascension $=a$, SD the declination $=\delta$, then applying Napier's rules to the right-angled triangle ♈ SD,

$$\tan a=\cos\omega.\tan l\ \ldots(1),$$

$$\tan\delta=\sin a\ .\tan\omega\ \ldots(2),$$

$$\sin\delta=\sin\omega.\sin l\ \ldots(3),$$

$$\cos l=\cos a.\cos\delta\ \ldots(4);$$

which equations, when two of the quantities a, ω, δ, l are given, determine the others.

79. Cor. 1. The equations (1) and (2) admit of useful transformations; substituting in (1) the imaginary values of tan a, tan l, and proceeding as in Art. 62, we obtain,

$$a = l - \tan^2 \tfrac{1}{2}\omega \,.\, \sin 2l + \tfrac{1}{2} \,.\, \tan^4 \tfrac{1}{2}\omega \,.\, \sin 4l - \&c.$$

$$l = a + \tan^2 \tfrac{1}{2}\omega \, \sin 2a + \tfrac{1}{2} \,.\, \tan^4 \tfrac{1}{2}\omega \,.\, \sin 4l + \&c.$$

$$\therefore\; l - a = \tan^2 \tfrac{1}{2}\omega \,.\, \sin 2l - \tfrac{1}{2} \,.\, \tan^4 \tfrac{1}{2}\omega \,.\, \sin 4l + \&c. \;.. \;(\alpha),$$

$$\text{and } l - a = \tan^2 \tfrac{1}{2}\omega \,.\, \sin 2a + \tfrac{1}{2} \,.\, \tan^4 \tfrac{1}{2}\omega \,.\, \sin 4a + \&c. \;.. \;(\beta).$$

These series are called the reductions of the ecliptic to the equator, and are useful in calculating tables.

80. Cor. 2. Proceeding in the same manner with equation (2), we have

$$\delta = \omega - \frac{1 - \sin a}{1 + \sin a} \,.\, \sin 2\omega + \&c.$$

$$\omega = \delta + \frac{1 - \sin a}{1 + \sin a} \,.\, \sin 2\delta + \&c.$$

$$\text{But, } \frac{1 - \sin a}{1 + \sin a} = \frac{1 - \cos \overline{90 - a}}{1 + \cos \overline{90 - a}} = \tan^2 \tfrac{1}{2} (90^0 - a)$$

$$= \tan^2 \tfrac{1}{2} u, \text{ (if } 90^0 - a = u);$$

$$\therefore\; \omega - \delta = \tan^2 \tfrac{1}{2} u \,.\, \sin 2\omega - \tfrac{1}{2} \,.\, \tan^4 \tfrac{1}{2} u \, \sin 4\omega + \&c. \;.. \;(\gamma),$$

$$\omega - \delta = \tan^2 \tfrac{1}{2} u \,.\, \sin 2\delta + \tfrac{1}{2} \,.\, \tan^4 \tfrac{1}{2} u \, \sin 4\delta + \&c. \;.... \;(\delta).$$

The series γ, δ are called the reductions of the declination to the solstitial declination.

81. The reduction of the declination to the solstitial declination, in terms of the longitude, may be deduced from the series (γ), (δ), or more readily from the equation

$$\sin \delta = \sin \omega \,.\, \sin l,$$

by Maclaurin's Theorem.

$$\text{Let } \delta = \omega - y, \ l = 90^0 - u;$$

$$\therefore \sin(\omega - y) = \sin\omega \,.\, \cos u$$

$$= \sin\omega - 2\sin\omega \,.\, \sin^2 \tfrac{1}{2} u.$$

$$\text{Let } 2\sin\omega \,.\, \sin^2 \tfrac{1}{2} u = x;$$

$$\therefore \sin(\omega - y) = \sin\omega - x;$$

$$\therefore \cos(\omega - y) \,.\, y' = 1,$$

$$\cos(\omega - y) \,.\, y'' + \sin(\omega - y) \,.\, y'^2 = 0,$$

$$\&\text{c.} \qquad\qquad = \&\text{c.};$$

making $x = 0$ in these equations, we obtain

$$y_0 = 0,$$

$$y_0{}' = \frac{1}{\cos\omega},$$

$$y_0{}'' = -\tan\omega \,.\, \frac{1}{\cos^2\omega};$$

therefore substituting these values in Maclaurin's theorem, and replacing x by its value $2\sin\omega \,.\, \sin^2 \frac{1}{2} u$, we have

$$y = 2\tan\omega \,.\, \sin^2 \tfrac{1}{2} u - 2\tan^3\omega \,.\, \sin^4 \tfrac{1}{2} u + \&\text{c.} \ldots (\epsilon).$$

In the equations α, β, γ, δ, ϵ the arcs ω, l, y, &c. are expressed in parts of the radius of the circle; expressed in seconds, the last equation becomes

$$y = \frac{2\tan\omega}{\sin 1''} \,.\, \sin^2 \tfrac{1}{2} u - \frac{2\tan^3\omega}{\sin 1''} \,.\, \sin^4 \tfrac{1}{2} u + \ldots$$

and the equations α, β, γ, δ may be reduced to seconds in the same way.

82. The formulæ γ, δ, ϵ, afford the most exact means of finding the obliquity. By Art. 71, a near value of ω may be

obtained, from which, and an observed declination of the Sun near the solstice, u may be found by means of the equation

$$\sin \delta = \sin \omega \,.\, \sin l, \ = \sin \omega \,.\, \cos u.$$

Substituting these values in the equation ϵ, a value of y is obtained, and if, from n declinations, δ, δ' near the solstices, n reductions y, y', &c. are deduced. The obliquity

$$= \frac{\delta + \delta' + \&c. + y + y' + \&c.}{n}, \text{ very nearly.}$$

The first term of the equation ϵ suffices for finding the reduction y, if the observations are made within 7 or 8 days of the Sun's being in the solstice; the observations may be extended through double that time, if the second term be made use of.

83. *To find the position of the ecliptic in the heavens at any time, with reference to the meridian and horizon of any place.*

Let ΥE (fig. 15 *) be the equator,

ΥO . . . . ecliptic,

HR . . . . horizon,

P, K, Z, their respective poles.

Let the meridian PZ cut ΥO, ΥE, and HR in C, c and H respectively, and let KZ cut ΥO in N, and HO in I; then C is called the *Culminating Point* of the ecliptic, or the *Mid Heaven*, O the *Ascending* point, and N the *Nonagesimal Degree*. Now the arc joining the poles of two great circles measures their inclination, and being produced, cuts the circles 90^0 from their point of intersection; therefore, ZK is the inclination of the ecliptic to the horizon, and $NO = 90^0$; for the same reason, OI, EH, ♈ T, ♈ t, all equal 90^0, T and t being the intersections of the equator and ecliptic with the solstitial colure.

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* In the figure, the arcs CO, HR, &c., are supposed to be on the eastern hemisphere, and $C\Upsilon$, $c\Upsilon$ on the west, so that Qc, or from the right hand side of the page to the left, is the direction of the diurnal motion, or from east to west, and cQ is the direction from west to east, or the order of the signs.

Let S be the place of the Sun referred to the equator, by a declination circle, then, the *right* ascension of the mid heaven $(A) = \text{♈} c = \text{♈} S + Sc =$ the Sun's right ascension + the Sun's distance from the meridian measured along the equator = the Sun's right ascension + the hour angle from noon..

Let $l = \text{♈} N$ be the longitude of the nonagesimal,

$a = HO$ ...... south azimuth of the ascending point.

Then, in the triangle ZPK, we have given

PK the obliquity $= \omega$,

PZ .. co-latitude $= c$,

$$ZPK = 180^0 - ZPT = 180^0 - (\text{♈} T - \text{♈} c) = 90^0 + A, ..$$

to find

$$ZKP = Nt = \text{♈} t - \text{♈} N \ldots\ldots\ldots\ldots = 90^0 - l,$$

$$PZK = CZN = HI \ldots\ldots\ldots\ldots\ldots\ldots = a - 90^0,$$

$ZK = NI$, the altitude of the nonagesimal;

therefore, by Napier's analogies,

$$\tan \tfrac{1}{2} . (a - l) = \frac{\cos \tfrac{1}{2} . (PZ - PK)}{\cos \tfrac{1}{2} . (PZ + PK)} . \cot \tfrac{1}{2} (90^0 + A),$$

$$\tan (90^0 - \tfrac{1}{2} (a + l)) = \frac{\sin \tfrac{1}{2} . (PZ - PK)}{\sin \tfrac{1}{2} . (PZ + PK)} . \cot \tfrac{1}{2} (90^0 + A),$$

$$\text{and} \quad \sin ZK = \sin PK \cdot \frac{\sin ZPK}{\sin PZK}$$

$$= \frac{\sin \omega \cdot \cos A}{-\cos a},$$

from which equations, a, l, ZK may be found.

The inclination of the ecliptic to the horizon, and the azimuth of the ascending point, are all that are required for ascertaining the position of the ecliptic with respect to the horizon, but l, the longitude of the nonagesimal, will be required in a subsequent Chapter.

84. Cor. When Aries rises, the solstitial colure coincides with the meridian, and therefore, K falls on the meridian, but since the Sun in moving eastward through ♈, passes into north declination that part of the ecliptic which lies to the east of ♈ lies north of the equinoxial; therefore, when ♈ rises in north latitudes, Υm the part of the ecliptic which is west of ♈, falls below the equator Υn, and K lies in PZ or PZ produced, and not on the other side of P. As ♈ rises, or moves in the direction $E\Upsilon$, K moves in the same direction, and the angle ZPK increases; therefore, since the sides ZP, PK, are constant, ZK, or the inclination of the ecliptic and horizon, increases continually, from the time when K is on the meridian, or ♈ rises, till $ZPK = 180^0$, when the solstitial colure again coincides with the meridian, and ♈ sets.

85. *Given the right ascension and declination of a star to find its latitude and longitude; and conversely.*

Let ΥD, (fig. 16.) the right ascension of the star S, $=a$,

SD . . . . declination . . . . $=\delta$,

ΥL . . . . longitude . . . . $=l$,

SL . . . . latitude . . . . $=\lambda$,

$L\Upsilon D$ . . . . the obliquity $=\omega$, and $S\Upsilon D = \phi$.

Then, in the right-angled triangle

$$S♈D,\quad \cot\phi = \cos a \,.\, \cot\delta \;\ldots\ldots\; (1),$$

$$\cos\phi = \tan a \,.\, \cot ♈S \;\ldots\; (2),$$

$$S♈L,\quad \cos(\phi - \omega) = \tan l \,.\, \cot ♈S \;\ldots\; (3),$$

$$S♈D,\quad \sin\delta = \sin\phi \,.\, \sin ♈S \;\ldots\; (4),$$

$$S♈L,\quad \sin\lambda = \sin(\phi - \omega) \,.\, \sin ♈S \;\ldots\; (5),$$

therefore, from (2) and (3),

$$\tan l = \frac{\cos(\phi - \omega) \,.\, \tan a}{\cos\phi}.$$

from (4) and (5), $\sin\lambda = \dfrac{\sin(\phi - \omega) \,.\, \sin\delta}{\sin\phi}$;

from which equations l and λ may be found by means of the subsidiary angle ϕ, obtained from equation (1).

86. The formula for finding the right ascension and declination from the latitude and longitude, may be deduced in exactly the same manner, but it is obvious that they may be derived from the former, by changing λ into δ, l into a, and making ω negative. They are, therefore,

$$\cot\phi = \cos l \,.\, \cot\lambda,$$

$$\tan a = \frac{\cos(\phi + \omega) \,.\, \tan l}{\cos\phi},$$

$$\sin\delta = \frac{\sin\lambda \,.\, \sin(\phi + \omega)}{\sin\phi},$$

$S♈L$ representing the subsidiary angle ϕ in this case.

82. Let OSL, the angle of position, $= S$,

$$OD = \theta; \;\therefore\; SO = (\delta - \theta).$$

Then in the triangle ΥOD, $\tan\theta = \sin a \,.\, \tan\omega \,\ldots\ldots\, (1)$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\sin\theta = \tan a \,.\, \cot \Upsilon OD \,..\, (2)$,

$\ldots\ldots\ldots\ldots\ldots\ldots$ SOL, $\cos(\delta - \theta) = \cot S \,.\, \cot \Upsilon OD \,..\, (3)$;

therefore, from (2) and (3),

$$\cot S = \frac{\cos(\delta - \theta) \,.\, \tan a}{\sin\theta},$$

which determines S from a and δ by means of the subsidiary angle θ.

87. Similarly, if the arc of SL *produced*, between the ecliptic and equator be taken for the subsidiary angle θ,

$$\tan\theta = \sin l \,.\, \tan\omega,$$

$$\cot S = \frac{\cos(\lambda + \theta) \,.\, \tan l}{\sin\theta},$$

which equations determine the angle of position, when the latitude and longitude of the star is given.

88. Produce DS, LS (fig. 17.) to meet the poles of the equator and ecliptic in P and K, and let KP meet the equator and ecliptic in Q and C; then, KP is the solstitial colure, and Υ is its pole, therefore, the arcs ΥC, ΥQ, ΥP, ΥK, each $= 90^0$, and also the angles ΥKP, ΥPK, &c.;

$$\therefore\ SPK = 90^0 + a,\ SKP = 90^0 - l,\ PK = \omega,$$

$$SK = 90^0 - \lambda,\ SP = 90 - \delta.$$

PSK, the angle of position of the star, $= S$.

$$\text{Hence } \cos SPK = -\sin a,\ \sin SPK = \cos a,$$

$$\cos SKP = \sin l,\ \sin SKP = \cos l,$$

and applying these values and the usual formulæ of Trigonometry to the triangle SPK, the following equations may be easily obtained:

$$\tan a = \cos\omega \,.\, \tan l - \tan\lambda \,.\, \sin\omega \,.\, \text{sect}\, l \,\ldots\, (1),$$

$$\sin\delta = \sin\omega \,.\, \cos\lambda \,.\, \sin l + \cos\omega \,.\, \sin\lambda \,\ldots\, (2),$$

$$\tan l = \cos\omega \,.\, \tan a + \sin\omega \,.\, \tan\delta \,.\, \text{sect}\, a \,\ldots\, (3),$$

$$\sin\lambda = \cos\omega \,.\, \sin\delta - \sin\omega \,.\, \cos\delta \,.\, \sin a \,\ldots\, (4),$$

$$\cos\delta \,.\, \cos a = \cos\lambda \,.\, \cos l \,\ldots\ldots\, (5),$$

$$\cot S = \cot\omega \,.\, \cos\delta \,.\, \text{sect}\, a + \tan a \,.\, \sin\delta \,\ldots\, (6),$$

$$= \cot\omega \,.\, \cos\lambda \,.\, \text{sect}\, l - \tan l \,.\, \sin\lambda \,\ldots\, (7).$$

The figure represents the first quadrant of the sphere; in the second quadrant, SKP would $= l - 90°$, and in the same way, that and SPK would alter in the other quadrants; but the algebraical results above deduced are of course general, and independent of the particular position of the geometrical figure whence they were derived.

89. We are now enabled to explain the method of making a catalogue of the fixed stars, and of registering their places either with respect to the ecliptic or the equator.

Having found (Art. 71.) the Sun's right ascension, let the hand of the sidereal clock be so placed, that the time by the clock, converted into degrees at the rate of 15^0 to one hour, may give the Sun's right ascension. Then, when Aries is on the meridian, the clock will point to 24^h, and if t^h be the time by the clock at any other instant, $15^0 . t$ will be the right ascension of all fixed stars then on the meridian, which may therefore be found. The declinations of the stars may be deduced, as that of the Sun was (Art. 69.), by subtracting the observed meridian zenith distances from the latitude, considering zenith distances, which are north of the zenith, negative, and all stars with negative declinations, to be south of the equator. When the stars are thus registered according to their right ascensions and declinations, a catalogue, in which they are ranged according to their latitudes and longitudes, may be formed from it by Art. 85. On account of the small changes of place of the fixed stars alluded to in Art. 68, the right ascensions, declinations, &c. observed at one time, will be different from those observed at another; but since the law of these changes is known, we are able, from the observations made at different periods, to calculate the right ascensions and declinations, such as they would have been found, if all the observations had been made at the same given epoch.

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CHAP. VI.

ON TIME.

90. Astronomers distinguish three kinds of days, the *Sidereal*, the *Solar*, and the *Mean solar*, each of which is divided into 24 equal parts or hours, each hour into 60 minutes, and each minute into 60 seconds.

91. The sidereal day is the interval between two successive transits of the first point of Aries (♈), and begins when that point is on the meridian.

The sidereal day is the same as the time of the Earth's rotation*, and may be considered equal to the time between two successive transits of the same fixed star over the meridian.

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* If ♈ were a fixed point, the sidereal day would be equal to the interval between two transits of the same fixed star; but the axis of the Earth, as will be noticed more fully in the Chapter on Precession, describes a conical surface round the axis of the ecliptic in 25680 years, in consequence of which, the intersection of the ecliptic and equator moves along the ecliptic through 50″.1, yearly, in a direction contrary to the order of the signs. This motion does not affect the length of the sidereal day, as defined in the text, which remains the same as if ♈ were a fixed point, because a given meridian, by the change in the position of the Earth's axis, is carried backwards as much as ♈, and the relative motion of ♈ and the meridian is not altered; but the interval between the successive transits of a fixed star will be changed, and will be less or greater than the time of the Earth's rotation, according as ♈ is carried to or from the star; that is, according as the star's right ascension is diminished or increased by precession. There are also other causes (Chap. x, xi.), which alter slightly the real or apparent right ascensions of the fixed stars, and therefore

A *Solar day* is the interval between two transits of the Sun over the same meridian, and begins at noon. This is sometimes called the *true* day, and sometimes the *apparent* day, because marked by a real phenomenon, the appearance of the Sun on the meridian, and not being, as the sidereal and mean solar days are, the results of calculation. The solar days, when compared by a good clock, are found to be unequal, which renders them unfit measures of time.

The *Mean Solar* day, as its name imports, is the mean of solar days, and is determined by the transits of an imaginary Sun, which describes the equinoxial with the Sun's mean motion in right ascension or longitude.

The difference between the true and mean time is called the *equation of time*, by the addition of which to true time, the mean time may be found. This quantity may be as much as $16' \ 17''$, or $-\ 14' \ 37''$; that is, a clock adjusted to mean time, may indicate $16' \ 17''$, or $23^{\mathrm{h}} \ 45' \ 23''$ when the true Sun is on the meridian. The means of determining the equation of time will be given in a subsequent Chapter; here it may be taken for granted, that when either the true or mean time is known, the other can be found, if the Sun's place in the ecliptic be given.

92. As the Earth revolves uniformly, a given meridian separates itself from the mean Sun by an angle of 360^0 in 24 mean solar hours; therefore, if h^0 be the distance of the mean Sun from the meridian, or the hour angle, $\frac{h}{15}$ will be the mean solar time in hours. In the same manner, if h^0 be the hour angle of the true Sun, or of ♈, $\frac{h}{15}$ will be the true, or sidereal time, respectively.

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therefore render unequal the intervals between the successive transits of different stars, or of the same star at different times. The difference, however, of a sidereal day, and of the interval between two successive transits of the same fixed star, except for stars very near the poles, is too small to be noted by the clock, and can only be detected in its accumulation for a long period.

93. As the Sun by its apparent motion from east to west moves round the Earth in 24 solar hours, if it be noon under a given meridian, the Sun will have been on a meridian 15° east of it, a *solar* hour before, and will pass a meridian 15° west an hour after; therefore it is one o'clock under the former, and eleven under the latter meridian. In the same manner it may be shewn, that for every additional 15° of east longitude, the clock is a *solar* hour faster, and for every 15° of west longitude, an hour slower. The same is true of the transits of the mean Sun, or of a fixed star, but the hours by which the clock is faster or slower, are, in the former case, mean solar hours, and in the latter, sidereal hours.

Cor. Any phenomenon, which is visible at the same instant of absolute time, will appear to happen an hour later for every 15° of east longitude, and an hour earlier for every 15° of west longitude; so that if the time of any observation is given, the absolute instant of its happening is not determined, unless the meridian from which the time is reckoned be given.

The time used in this work is always mean time, unless the contrary be expressed, and is that to which clocks and watches are usually adapted. Astronomical clocks are often regulated to sidereal time. The hours pointed out by sun dials are in solar time.

94. The interval from the Sun leaving Aries, till it returns to it again, is $365^d\ 5^h\ 48'\ 51''$, in which time the Sun separates from Aries by an angle of 360°; therefore the Sun's mean motion in longitude in one mean solar day $= 59'\ 8''.33$, which is obtained by dividing the whole angle 360° by the whole time, $355^d\ 5^h\ 48'\ 51''$.

95. The mean Sun in the equator, which moves with the Sun's mean motion in longitude, describes an angle of $59'\ 8''.33$ in a mean solar day; therefore in the interval between two successive transits over the same meridian, the Earth must revolve through $360°\ 59'\ 8''.33$; but in a sidereal day the Earth revolves through 360°;

$\therefore$ a sidereal day : a mean solar day :: $360° : 360°\ 59'\ 8''.33$

96. *To convert sidereal into mean solar time, and conversely.*

Since the number of hours, minutes, or seconds, in any given interval of absolute time must be inversely as the length of one; therefore, if m be any time, expressed in mean hours, minutes and seconds, s the same time expressed in sidereal hours, minutes and seconds,

$$m : s :: 360^0 : 360^0\ 59'\ 8''.33;$$

$$\therefore m = s \,.\, \frac{360^0}{360^0\ 59'\ 8''.33} = s - \frac{59'\ 8''.33}{360^0\ 59'\ 8''.33} \,.\, s \,.\,.\, (1),$$

$$s = m \,.\, \frac{360^0\ 59'\ 8''.33}{360^0} = m + \frac{59'\ 8''.33}{360^0} \,.\, m \,.\,.\,.\,.\,.\, (2).$$

By equation (1), any period, expressed in sidereal time, may be reduced to mean, and the converse, by equation (2).

97. Cor. These reductions may be facilitated, by calculating the quantities to be added or subtracted for one day, one hour, and one minute.

To reduce any time expressed in sidereal hours, minutes and seconds, to mean hours, &c.

For each day subtract $\frac{59'\ 8''.33}{360^0\ 59'\ 8''.33}$ days $= 3'\ 55''.9093$,

.... hour ...... $\frac{3'\ 55''.9093}{24}$ .... $=$ $9''.8296$,

...... minute ..... $\frac{9''.8296}{60}$ ..... $=$ $0''.1636$,

..... second .... $0''.00273$.

Similarly, any period, expressed in mean time, may be reduced to sidereal, by adding $\frac{59'\ 8''.33}{360^0}$. days, or $3'\ 56''.5553$ for each day, $9''.8568$ for each hour, $0''.1643$ for each minute, and $0'',0273$ for each second.

Ex. Express the sidereal interval $10^h\ 8'\ 40''$ in mean time.

$$
\begin{array}{lrr}
 & \text{The sidereal interval} & = 10^h\ 8'\ 40''. \\
\text{For } 10^h \text{ subtract} & 10\,.\,(9''.819) = & 1'\ 38''.29, \\
8' \ldots\ldots & 8\,.\,(0''.163) = & 1''.30, \\
40'' \ldots\ldots & 40\,.\,(0''.0027) = & 0''.10. \\
\end{array}
$$

Therefore the interval in mean time $= 10^h\ 7'\ 0''.1$.

98. Cor. A sidereal day $= 24^h - (3'\ 55''.9093) = 23^h\ 56'\ 4''$ mean time; and a mean solar day $= 24^h\ 3'\ 56''$ sidereal time: a clock which marks either may be regulated to the other by a proper adjustment of the pendulum.

99. *Given the sidereal time of any phenomenon, under a given meridian, to find the corresponding mean solar time, and conversely.*

Let $AM♈$ (fig. 18.) be the equator, M the place of the mean Sun, A of the meridian, ♈ the first point of Aries, a the mean Sun's right ascension at the preceding mean noon, or when M was at A, b the increase in 24 mean solar hours; and let $A♈$, $M♈$, a and b be all reduced to time;

$$\therefore\ b = \frac{59'\ 8''.33}{15^0}\,.\,24^h = 3'\ 56''.55.$$

Let m be the mean time,

s . . . . . sidereal . . . .;

$$\therefore\ s = A♈ = M♈ + AM = M♈ + m.$$

But $M♈ = a +$ the increase of M's right ascension in the time from A to M, (or in the time m) $= a + b\,.\,\frac{m}{24}$;

$$\therefore\ s = a + m + \frac{b}{24}\,.\,m \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1);$$

$$\therefore\ m=\overline{s-a}\,.\,\frac{24}{24+b}=\overline{s-a}-\overline{s-a}\,.\,\frac{b}{24+b}\ .\ .\ .\ (2).$$

Ex. Given the sidereal time $= 5^{\mathrm{h}}\ 57'\ 56''$, to find the mean time, supposing the mean Sun's right ascension when on the meridian $= 1^{\mathrm{h}}\ 17'\ 6''$,

$$s = 5^{\mathrm{h}}\ 47'\ 56'',$$

$$a = 1^{\mathrm{h}}\ 17'\ 6'';$$

$$\therefore\ s-a = 4^{\mathrm{h}}\ 30'\ 50''.$$

For 4^{h} subtract $39''.3$,

$30'$.......... $4''.9$,

$50''$......... $0''.1$,

$44''.3$,

$4^{\mathrm{h}}\ 30'\ 50''$.

$4^{\mathrm{h}}\ 30'\ \ 5''.7$ = the mean time.

100. Cor. 1. To find the mean time of a star's transit.

Since the star is then on the meridian, s = the star's right ascension in time $= a$;

$$\therefore\ m=a-a-(a-a)\,.\,\frac{b}{24+b},$$

which is the mean time of the star's transit.

101. Cor. 2. To find the mean time of a planet's transit.

Let A (fig. 18.) be the planet on the meridian, P the right ascension at the preceding mean noon, e the increment of right ascension in 24 mean hours, both expressed in time.

Then $A\Upsilon = P +$ increment of P in the time m,

$$=P+e.\frac{m}{24},$$

and $$M\Upsilon=a+b.\frac{m}{24};$$

therefore, subtracting,

$$AM \text{ or } m=P-a+\overline{e-b}.\frac{m}{24};$$

$$\therefore m=\frac{(P-a).24}{24-e+b}$$

$$=P-a+\frac{e-b}{24-e+b}.(P-a).$$

If the body's motion is retrograde, e is negative.

102. *To find the hour angle described by a planet in a given interval of mean solar time.*

Let b^0 (59′ 8″.33) be the mean Sun's motion in Æ in 24 hours,

n^0 . . . . planet's . . . .

Then, if the planet and Sun be together on the same meridian, when the Sun is next on the meridian, the planet will be at a distance n^0-b^0, or, instead of having separated from the meridian by 360^0 will have separated only by $360^0-n^0+b^0$, in 24 mean solar hours; therefore, if T be the time corresponding to an hour angle P,

$$T:24::P:360^0-n^0+b^0;$$

$$\therefore P=\frac{T}{24}.(360^0-n^0+b^0).$$

103. Cor. 1. $T=\frac{P}{360^0-n^0+b^0}.24^h$, in mean solar hours.

104. Cor. 2. The time between two successive transits of a planet

$$= \frac{360^0}{360^0 - n^0 + b^0} \,.\, 24^h.$$

To determine the time by observation.

105. 1st method. *By observing the time of transit of the Sun, or a known star.*

If the Sun be the body observed, the time of transit will be the apparent noon, and the equation of time being added, will give the corresponding mean time. If a star be the body observed, its right ascension converted into time, will be the sidereal time from whence the mean time may be found.

106. 2d method. *By an observed altitude of the Sun.*

Let z be the Sun's zenith distance $= 90^0 -$ the observed altitude,

Δ the north polar distance,

h the hour angle,

c the co-latitude.

Then in the triangle PZS, (fig. 10.)

$$\sin \tfrac{1}{2} . h = \sqrt{\frac{\sin \frac{1}{2}(z+\Delta-c) \,.\, \sin \frac{1}{2} . (z+c-\Delta)}{\sin c \,.\, \sin \Delta}};$$

whence h, and therefore $\frac{h}{15}$, the apparent time, may be found.

If z be the zenith distance, and Δ the north polar distance of a star, whose right ascension is a, $\frac{a \pm h}{15}$ is the sidereal time; $+$ or $-$ according as the star is west or east of the meridian.

107. *Given a small error in the observed altitude, to find the corresponding error in the time computed from it.*

Retaining the same notation;

$$\text{since } \cos h = \frac{\cos z - \cos c . \cos \Delta}{\sin c . \sin \Delta};$$

$$\therefore \sin h . dh = \frac{\sin z}{\sin c . \sin \Delta} . dz$$

$$= \frac{\sin h}{\sin c . \sin A} . dz.$$

A being the azimuth;

$$\therefore dh = \frac{dz}{\sin c \sin A}.$$

Now when the errors are very small, dh and dz are the corresponding errors of the hour angle and zenith distance;

$$\text{therefore the error in time} = \frac{1}{15^\circ} . \frac{1}{\sin c . \sin A} . \text{error in altitude.}$$

Cor. Hence, the error in time, from a given error in altitude, is less the greater sin A is, or the nearer the body is to the prime vertical, at the instant of observation.

108. In order to avoid instrumental errors, it is usual, in taking altitudes with multiplication circles, to take several at equal very short intervals, and to consider the mean altitude, or mean zenith distance as corresponding to the mean of the times of observation. This supposes the change of altitude to vary as the time, which is not exactly the case.

Let t be the time, in degrees, corresponding to the zenith dist. z,

t' .... z',

t'' .... $\frac{z+z'}{2}$.

Then, since $z'=z+(z'-z)$, and $\frac{1}{2}.(z'+z)=z+\frac{1}{2}.(z'-z)$;

$$\therefore\ t'=t+\frac{dt}{dz}.(z'-z)+\frac{d^2t}{dz^2}.\frac{(z'-z)^2}{1.2}+\&c. \ldots (1),$$

$$t''=t+\frac{dt}{dz}.\ \frac{z'-z}{2}\ +\frac{d^2t}{dz^2}.\frac{(z'-z)^2}{8}+\&c. \ldots (2);$$

therefore dividing (1) by 2, and subtracting it from (2),

$$t''=\tfrac{1}{2}.(t+t')-\frac{d^2t}{dz^2}.\frac{(z'-z)^2}{8},\ \text{nearly}.$$

But from (1) $z'-z=(t'-t).\frac{dz}{dt}$, nearly;

$$\therefore\ t''=\tfrac{1}{2}.(t+t')-\frac{(t'-t)^2}{8}.\frac{dz^2}{dt^2}.\frac{d^2t}{dz^2}.$$

But if h be the hour angle in degrees,

$$dt=dh,\ \text{which}=\frac{dh}{\sin 1''},$$

when h and dh are expressed in parts of radius;

$$\therefore\ \frac{d^2t}{dt^2}=\sin 1''.\frac{d^2h}{dh^2};$$

$$\therefore\ t''=\tfrac{1}{2}.(t+t')-\frac{(t'-t)^2}{8}.\sin 1''.\frac{d^2h}{dh^2},$$

where h is expressed in parts of radius.

But $\cos z=\cos l.\cos\delta.\cos h+\sin l.\sin\delta$;

$$\therefore\ \frac{dz}{dh}=\frac{\cos l.\cos\delta.\sin h}{\sin z};$$

$$\therefore\ -\frac{d^2h}{dh^2}=\frac{\cos l.\cos\delta}{\sin^2 z}.\left(\sin z.\cos h.\frac{dh}{dz}-\sin h.\cos z\right)$$

$$= \frac{\cos l \,.\, \cos \delta}{\sin^2 z} \,.\, \left(\frac{\sin z \,.\, \cos h \,.\, \sin z}{\cos l \,.\, \cos \delta \,.\, \sin h} - \sin h . \cos z \right)$$

$$= \cot h - \frac{\sin h \,.\, \cos z \,.\, \cos l \,.\, \cos \delta}{\sin^2 z};$$

and calling this quantity $-M$,

$$t'' = \tfrac{1}{2} \,.\, (t + t') - \tfrac{1}{8} \,.\, (t' - t)^2 . \sin 1'' . M.$$

Let therefore Z be the sum of the zenith distances, T of the times, n the number of observations, θ the interval between the first and last, θ' between the second and last but one, and so on; then, the time t, corresponding to the zenith distance $\frac{Z}{2n}$,

$$= \frac{T}{2n} - \frac{1}{8n} \,.\, (\theta^2 + \theta'^2 + \&c.) \,.\, \sin 1'' \,.\, M.$$

When the body is on the prime vertical, $\frac{dt}{dz}$ is a minimum, and therefore $\frac{d^2 t}{dz^2} = 0$. Hence, when the observations are made very near the prime vertical,

$$t = \frac{T}{2n}.$$

109. 3d. method. *By equal or corresponding altitudes of the Sun or a known star, taken before and after culmination.*

Let the star be observed at equal altitudes on each side of the meridian, then, since the declination is invariable, the corresponding hour angles are equal, and the time by the clock, of the star's transit = half the sum of the times of observation; if this time be compared with the known sidereal time of the star's transit (Art. 100.), the deviation of the time marked by the clock from sidereal time, at the instant of culmination is obtained.

The time, by the clock, of the star's culmination, will be less liable to be vitiated by errors of observation, if several pairs of observations are made, at equal altitudes, on each side of the meridian, and if the sum of all the times, divided by twice the number of observations, be taken for the time of the star's transit.

110. If the deviation of the clock from sidereal time be found after one or more days, the rate, or daily gain of the clock over sidereal time is ascertained. If the clock gain e'' in 24 sidereal hours, in t^h the gain is $\frac{t}{24} . e''$, and if τ^h be the interval by the clock, corresponding to an interval of t sidereal hours,

$$t = \tau - \frac{t}{24} . e'';$$

$$\therefore\ t = \frac{24^h}{24^h + e''} . \tau = \frac{86400}{86400 + e} . \tau.$$

If the clock loses with respect to sidereal time, e is negative.

Cor. Hence, if h^0 be the hour angle described in τ^h by the clock,

$$h^0 = 15^0 . t = 15^0 . \frac{86400}{86400 + e} . \tau,$$

$$\text{and } \tau = \frac{86400 + e}{86400} . \frac{h}{15}.$$

In exactly the same manner, as in the last two Articles, the deviation of a clock from *mean* or *apparent* time may be ascertained, and the *mean* or apparent interval corresponding to any interval by the clock.

111. When equal altitudes of the Sun are taken before and after noon, the time of transit is not exactly the semi-sum of the times of observation, on account of the Sun's change of declination. To correct for this, let h and h' be the hour angles,

δ and δ' the declinations of the Sun, corresponding to the common zenith distance z;

$$\therefore \cos z = \cos l . \cos \delta . \cos h + \sin l . \sin \delta.$$

Then, since z and l remain the same whilst h and δ change, differentiating with respect to these quantities alone, we have

$$0 = (\sin l \cos \delta - \cos l \cos h \sin \delta)\, d\delta - \cos l . \cos \delta . \sin h . dh,$$

but $d\delta$ and dh = the cotemporary increments $h' - h$ and $\delta' - \delta$ respectively;

$$\therefore h' - h = \frac{\tan l - \cos h . \tan \delta}{\sin h} . (\delta' - \delta);$$

$$\therefore \frac{1}{2} . \frac{h' - h}{15^0} = \frac{\tan l - \cos h \tan \delta}{\sin h} . \frac{\delta' - \delta}{30^0};$$

and subtracting this quantity, which is expressed in parts of an hour, from the semi-sum of the times of observation, the time by the clock of the Sun's transit is obtained, supposing the clock to be adjusted to apparent time.

If the interval, by the clock, between the preceding and succeeding noons, is $24^{\text{h}} + e''$, and not 24^{h}, the correction is

$$\frac{1}{2} . \frac{86400 + e}{86400} . \frac{h' - h}{15}$$

$$= \frac{1}{2} . \frac{86400 + e}{86400} . \frac{\tan l - \cos h . \tan \delta}{\sin h} . \frac{\delta' - \delta}{15} .$$

which must be subtracted from the semi-sum of the times of observation, as before.

112. 4th method. *From two altitudes of the Sun, or of a known star, and the time between.*

This method, which is adopted only when the latitude is unknown, has been already investigated in Art. 50, where the latitude and hour angle were found from two altitudes, and the time between; having found the hour angle, the apparent time is known, if the Sun be the body observed, and the sidereal time, if the body observed be a known star.

113. When the latitude and time are determined from two altitudes and the time between, to find the errors caused by given small errors in the observed altitudes.

Let P (fig. 19.) be the pole, Z the zenith, S the Sun at the first observation, Z' and S' the places of Z and S determined from the erroneous altitudes; then since the Sun's declination is supposed to be known without error, $PS = PS'$, and SS' is a parallel of declination. Let dl, dz, dz', dh, dh', be the cotemporary errors of l, z, z', h and h', where, since the interval $h'-h$ between the observations is supposed known correctly, $dh' = dh$, then drawing Zm, Sn, perpendiculars to $Z'S'$, and making $ZZ'S$ the south azimuth of $S = a$, we have

$$\begin{aligned} dz &= ZS - Z'S' = Zm + Sn, \\ &= dl \,.\, \cos a + SS' \,.\, \cos S'Sn, \end{aligned}$$

$$\begin{aligned} \text{but } SS' \,.\, \cos S'Sn &= \sin PS \,.\, dh \,.\, \sin ZSP \\ &= dh \,.\, \sin PZ \,.\, \sin PZS \\ &= dh \,.\, \cos l \,.\, \sin a; \end{aligned}$$

$$\therefore\ dz = dl \cos a + dh \,.\, \cos l \,.\, \sin a.$$

Similarly, if a' be the azimuth at the second observation,

$$dz' = dl \cos a' + dh \,.\, \cos l \sin a';$$

$$\therefore\ dl = \frac{dz \sin a' - dz' \sin a}{\sin (a' - a)};$$

$$dh = \frac{dz' \cos a - dz \cos a'}{\cos l \,.\, \sin (a' - a)}.$$

Hence, in determining the latitude and time from two altitudes and the time between, $a' - a$ ought to be as near 90^0 as possible, so that if the one observation is made near the prime vertical, the other must be made near the meridian.

114. The four methods which have been described are those most frequently adopted in finding the time by observation; the three last may be applied generally, by sea or land, but the first, in which a transit instrument is required, can only be made use of in a fixed Observatory. There is a great variety of other ways in which the time may be found from observations; and as the investigation in general is attended with no difficulty, it will be sufficient here to give two of them, which can be most frequently applied.

115. *To find the time from observing when two known stars have the same altitude, the altitude itself being unknown.*

Let a, a' be the right ascensions of the two stars, δ, δ' their declinations, h, h' the hour angles, z the common zenith distance, and l the latitude of the place.

Then, if the stars are observed at the same instant,

$$h' - h = a' \sim a.$$

If they are observed after an interval of t sidereal hours,

$$h' - h = a' \sim a \pm 15^0 t \text{ (Art. 52.)}.$$

If this quantity $= \alpha$, $h' = h + \alpha$, and we have

$$\cos z = \cos l \,.\, \cos \delta \,.\, \cos h + \sin l \,.\, \sin \delta,$$

$$\cos z = \cos l \,.\, \cos \delta' \,.\, \cos (h + \alpha) + \sin l \,.\, \sin \delta'$$

$$= \cos l \,.\, \cos \delta' \,.\, \cos \alpha \,.\, \cos h - \cos l \,.\, \cos \delta' \sin \alpha \sin h + \sin l \,.\, \sin \delta';$$

therefore, subtracting

$$0 = \cos l \,.\, (\cos \delta' \cos \alpha - \cos \delta) \cos h$$

$$- \cos l \,.\, \cos \delta' \sin \alpha \sin h + \sin l \,.\, (\sin \delta' - \sin \delta);$$

therefore,

$$(\cos \delta' \cos \alpha - \cos \delta) \,.\, \cos h - \cos \delta' \,.\, \sin \alpha \,.\, \sin h = \tan l \,.\, (\sin \delta - \sin \delta').$$

Let ϕ be determined from the expression

$$\cot\phi = \frac{\cos\delta' . \cos a - \cos\delta}{\cos\delta' \sin a};$$

$$\therefore \cos\delta' . \sin a . (\cot\phi \cos h - \sin h) = \tan l . (\sin\delta - \sin\delta');$$

$$\therefore \cos\delta' . \sin a . \frac{\cos(\phi + h)}{\sin\phi} = 2 \tan l . \sin\tfrac{1}{2}(\delta - \delta') . \cos\tfrac{1}{2} . (\delta + \delta');$$

$$\therefore \cos(\phi + h) = \frac{2 \sin\phi . \tan l}{\cos\delta' . \sin a} . \sin\tfrac{1}{2}(\delta - \delta') . \cos\tfrac{1}{2} . (\delta + \delta'),$$

from which h, and therefore the time may be obtained.

116. *To find the time when two known stars are on the same vertical.*

Let Z (fig. 20.) be the zenith, P the pole, S and S' the two stars in the vertical ZS; draw Pp perpendicular to ZS, and let δ and δ' be the declinations of the stars, a and a' their right ascensions, and h the hour angle ZPS; then $SPS' = a' - a$, and if $SPp = \phi$, $S'Pp = a' - a - \phi$; therefore, applying Napier's rules to the triangles SPp, $S'Pp$,

$$\cos\phi = \tan\delta . \tan Pp \ldots\ldots\ldots (1),$$

$$\cos(a' - a - \phi) = \tan\delta' . \tan Pp \ldots\ldots\ldots\ldots (2);$$

$$\therefore \frac{\cos(a' - a - \phi) - \cos\phi}{\cos(a' - a - \phi) + \cos\phi} = \frac{\tan\delta' - \tan\delta}{\tan\delta' + \tan\delta}.$$

$$\text{Or, } \tan\tfrac{1}{2} . (a' - a) . \tan\tfrac{1}{2} . (2\phi - a' + a) = \frac{\sin(\delta' - \delta)}{\sin(\delta + \delta')};$$

$$\therefore \tan\tfrac{1}{2} . (2\phi - a' + a) = \frac{\sin(\delta' - \delta)}{\sin(\delta' + \delta)} . \cot\tfrac{1}{2}(a' - a),$$

which determines ϕ.

Again, in the right-angled triangle ZPp,

$$\cos(h+\phi) = \tan l . \tan Pp$$
$$= \tan l . \cos\phi . \cot\delta \text{ from (1)},$$

from which h and therefore the time of the observation may be found.

The instant when the stars are in the same vertical may be easily ascertained by suspending a plumb-line, and observing when they both cross the line at the same time. The pole star ought to be one of the stars chosen for this observation, because, on account of the small north polar distance, a star which is in the same vertical with the pole star must be near the meridian, and, as its motion is then nearly horizontal, the time when it crosses the vertical line can be noted more exactly.

117. COR. 1. If Z be the azimuth of the vertical plane in which the stars are together,

$$\sin l = \cot Z . \cot(h+\phi);$$
$$\therefore \cot Z = \sin l . \tan(h+\phi).$$

COR. 2. If S and S' be two positions of the *same* circumpolar star, observed after an interval of $2t$ sidereal hours; $SPS' = 15^\circ . 2t$; $\therefore \phi = 15^\circ . t$, and the equation for determining ϕ may be dispensed with. In this case, the vertical plane may be formed by suspending two plumb-lines.

118. By whatever method the time, whether mean or sidereal, is found, if it be compared with that shewn by the clock, the quantity by which the clock is fast or slow with respect to mean or sidereal time, may be ascertained, and also the daily gain, as in Art. 110.

The rate of a common watch may be easily found by observing with it the instant of disappearance of a bright star behind a building, or any other fixed object. This takes place always at the same point, after an interval of $23^h\ 56'\ 4''$, or $3'\ 56''$ earlier every night, by a watch regulated to mean time; therefore, by observing on different nights the times of disappearing of several bright stars, the watch's rate may be ascertained and corrected. This method is available where there is no telescope, the only precaution required being for the eye of the observer to be placed as near as possible in the same place, and to select stars which have small declinations, in order that the time of their disappearance may be noted more readily.

DEF. The *Tropical Year* is the time between the Sun's leaving and returning to the first point of Aries.

119. *To find the length of the tropical year.*

If the Sun be on any meridian when in ♈ at the beginning of the year, after 365 transits it will not have reached ♈, and after 366 will have passed it; therefore the tropical year consists of something more than 365^d. The length may be found most accurately by comparing observations made at distant periods. The Sun's right ascension can be found from his observed declination, and by comparing the right ascension when the Sun was near ♈ with observations made 36525 days, or nearly 100 years before, Delambre found the Sun's motion in right ascension in that time to be $100 . 360^0\ 45'\ 45''$. In one year the Sun separates from ♈ by 360^0,

$$\therefore\ 100 . 360^0\ 45'\ 45'' : 360^0 :: 36525^d : 365^d\ 5^h\ 48'\ 51''.6,$$

the length of a tropical year.

120. The year thus found is the mean tropical year, as, on account of irregularities in the motion of the Sun and of ♈, the tropical years differ a little from each other. The tropical year at any time may be obtained by finding A°, the increase of the Sun's right ascension in 365 days, from the proportion.

$$A^0 : 360^0 :: 365^d : \text{length of the tropical year.}$$

121. The correct value of the mean tropical year is $365^d.242264$. This length is not convenient for civil purposes, for which it is requisite that the year should consist of an exact number of days; to insure which, and at the same time to make the Sun, on whose declination the seasons depend, be near the vernal equinox at the commencement of the civil year, Julius

Cæsar made every 4th year to consist of 366^d, and other years of 365^d. This supposes 4 tropical years to contain $4.365+1$ days, or one year to consist of $365^d.25$, which exceeds the true value by $0^d.007736$, so that in 1000 years the civil year would begin more than 7^d after the Sun had passed the equinox. To correct this accumulating error, Pope Gregory made common years to consist of 365^d, and intercalated one day every fourth year as before, but omitted the intercalation in those centuries which are not divisible by 4, or only intercalated 97^d in 400^y instead of 100^d, as in the Julian correction. This supposes the length of one year to be $365^d + \frac{97}{400} = 365.2425$, which is too great by .000236, but the error does not amount to a day in 4500^y.

If the Gregorian intercalation of 97^d in 400^y, or 970^d in 4000^y be omitted every 4000th year, the civil year would be $365^d + \frac{969}{4000} = 365.242250$, and it would require more than 70000 years to make the beginning of the civil and tropical year be a day apart.

The Gregorian intercalation is adopted in most Christian countries, and is made on the 29th of February, which has 28^d in common years, and 29^d in leap years.

DEFINITION. A *Sidereal Year* is the time between the Sun's leaving and returning to the same position in the heavens.

122. *To find the length of a sidereal year.*

Since Aries moves with a slow annual motion of $50''.1$ to meet the Sun, the sidereal is greater than the tropical year, by the time the Sun takes to move over this space;

$$\therefore \text{ a sidereal year} : 365^d\ 5^h\ 48'\ 51''.6 :: 360^0 : 360^0 - 50''.1;$$

$$\therefore \text{ a sidereal year} = 365^d\ 6^h\ 9'\ 11''.5.$$

DEF. The Sun, as will be shewn hereafter, appears to describe an ellipse round the Earth, in which the point where it is nearest to the Earth is called the *Perigee*, and the point where it is most distant, the *Apogee*. The interval between two successive passages of the Sun through the apogee, is called an *Anomalistic Year*.

123. *To find the length of an anomalistic year.*

The apogee is not fixed, but moves through $11''.8$ in the direction of the Sun's motion in the course of a year; therefore, the anomalistic year is longer than the sidereal, by the time which the Sun takes to move over this space;

$\therefore$ the anomalistic year : $365^d\ 6^h\ 9'\ 11''.5 :: 360^0\ 11''.8 : 360^0$;

$\therefore$ the anomalistic year $= 365^d\ 6^h\ 13'\ 58''.8$.

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CHAP. VII.

ON THE TIMES OF RISING AND SETTING OF THE HEAVENLY BODIES.

124. *To find the length of the day at any place.*

Let Z (fig. 10.) be the zenith, P the pole, S the Sun at rising; then, $ZS = 90^0$, and ZPS is the hour angle from sunrise to mid-day. It is evident that the spherical triangle formed by P, Z and S at sun-set, on this side of the meridian, will be the same as PZS, if PS be constant; therefore, if $2t$ be the length of the day in solar hours,

$$ZPS = 15^0 . t.$$

Let l be the latitude $= 90^0 - PZ$,

δ the Sun's declination $= 90 - PS$;

$$\therefore \cos 15^0 . t = \frac{\cos 90^0 - \sin l . \sin \delta}{\cos l . \cos \delta} = - \tan l . \tan \delta.$$

Cor. Hence, if 2τ be the duration of the night in solar hours, since $2\tau = 24 - 2t$,

$$\cos 15^0 . (12 - \tau) = - \tan l . \tan \delta,$$

$$\text{or } \cos 15^0 . \tau = \tan l . \tan \delta.$$

125. If PS be not constant, the hour angles for the morning and evening are unequal, and therefore the times; hence, if t be the length of the morning and t' of the evening, δ the Sun's declination at sun-rise, and $(\delta + \Delta)$ at sun-set, where Δ is very small,

$$\begin{aligned}\cos 15t &= -\tan l . \tan \delta,\\ \cos 15t' &= -\tan l . \tan (\delta + \Delta)\\ &= -\tan l . \tan \delta - \tan l . \sec^2 \delta . \Delta, \text{ nearly}\\ &= \cos 15t - \tan l . \sec^2 \delta . \Delta.\end{aligned}$$

Therefore, if Δ be positive, or the declination increases, the evenings are longer than the mornings, if Δ be negative, they are shorter, and at the solstices, when $\Delta = 0$, the mornings and evenings are equal.

126. *To compare the lengths of the day at different times of the year.*

Suppose the observer in north latitude, and $2t$ as before the length of the day; therefore,

$$\cos 15^0 . t = - \tan l . \tan \delta.$$

$$\text{When } \delta = 0, \ \cos (15^0 . t) = 0; \ \therefore 2t = 12^h,$$

or the days and the nights are equal in all latitudes; as δ increases, $\cos 15^\circ t$ increases, till the Sun reaches his greatest north declination $23^0 . 28'$, at the time of the summer solstice, when the equation

$$\cos 15^0 . t = - \tan l . \tan 23^0 . 28'$$

determines the duration of the longest day. As δ decreases from $23^0\ 28'$, t diminishes, and when the Sun again crosses the equator in Libra, at which time $\delta = 0$, the days are again 12^h long all over the world.

After passing the equator δ is negative, and the equation becomes

$$\cos 15^\circ t = \tan l . \tan \delta;$$

therefore, as δ increases t diminishes, and when $\delta = 23^0\ 28$, the equation

$$\cos 15t = \tan l . \tan 23^\circ\ 28',$$

determines the length of the shortest day. As the Sun moves from the winter solstice to the vernal equinox, δ diminishes from $23^0\ 28'$ to 0, and the days increase in length, till at the vernal equinox the day again is 12 hours long.

If the Sun's declination = the co-latitude of the place of observation,

$$\cos 15t = -\tan l . \tan \delta = -1;$$

therefore, the day is 24^h long, and the Sun does not set during the day.

Hence, in latitude $66^0\ 32'$ the Sun does not set for 24^h, when at his greatest north declination $23^0\ 28'$, and in the higher latitudes the Sun will not set from the time when his declination = the co-latitude, till it again becomes equal to the same quantity after passing through its maximum $23^0\ 28'$.

In the same way it may be shewn that the Sun does not rise in latitude $66^0\ 32'$ for 24^h at the time of the winter solstice, and continues beneath the horizon when south of the equator, at all higher latitudes for the same time he was above the horizon when in north declination.

The preceding observations, applied to places in the southern hemisphere, shew that the length of the day there increases with the Sun's southern, and diminishes with his northern, declination, in such a manner that the length of the day at any time, at a place in north latitude, = the length of the night in the same latitude south.

The small circles of the sphere, $23^0\ 28'$ distant from the north and south poles, are called the *Arctic* and *Antarctic* circles.

DEF. A star is said to rise *heliacally* when the Sun is sufficiently below the horizon to admit of the stars being seen at rising. This depends on the magnitude of the Star, the clearness of the atmosphere, and other circumstances, but in general for stars of the first magnitude, the distance of the Sun below the horizon lies between 10^0 and 12^0.

A star rises *cosmically*, which rises at the same time with the Sun.

127. *To find when a given star rises* heliacally.

Let ΥS (fig. 21.) be the ecliptic, ΥQ the equator, Rs the horizon of the place of observation; therefore $\Upsilon Qs =$ the co-latitude; let s be a star rising heliacally, ΥT, Ts, its right ascension and declination, and S the Sun 10° below the horizon. Then, in the triangle TsQ,

$$\sin TQ = \tan \text{latitude} \times \tan \text{star's declination},$$

which determines TQ; $\therefore$ $\Upsilon Q =$ the star's right ascension $\pm$ TQ is known.

Hence, in triangle ΥQO, we know

$$\Upsilon QO = 90^0 + l,\ Q\Upsilon O = \omega, \text{ and } \Upsilon Q;$$

therefore ΥO and QO may be found from the equations

$$\tan \tfrac{1}{2}.(\Upsilon O + QO) = \frac{\cos \frac{1}{2}.(Q - \Upsilon)}{\cos \frac{1}{2}.(Q + \Upsilon)}.\tan \tfrac{1}{2}.\Upsilon Q,$$

$$\tan \tfrac{1}{2}.(\Upsilon O - QO) = \frac{\sin \frac{1}{2}.(Q - \Upsilon)}{\cos \frac{1}{2}.(Q + \Upsilon)}.\tan \tfrac{1}{2}.\Upsilon Q.$$

Hence, $\sin O = \sin Q.\dfrac{\sin \Upsilon Q}{\sin \Upsilon O}$ is known; but from the right-angled triangle SOR, $\sin 10^0 = \sin SO.\sin O$, which determines SO; and hence we know $\Upsilon S = \Upsilon O + OS =$ the Sun's longitude, from whence, as will be shewn, the time of the year may be found.

128. If s be a star rising *cosmically*, O is the place of the Sun, and ΥO, the Sun's longitude, may be found as before.

129. A great variety of problems relating to azimuths, altitudes, &c. may be solved by means of the triangle PZS, in

which P is the pole of the equator, Z the zenith, and S a heavenly body; if in this triangle we call

ZP the co-latitude $=c$, or 90^0-l,

PS .. co-declination of $S=\Delta$, or $90^0-\delta$,

ZS .. zenith distance of $S=z$,

ZPS .. hour angle $=h$, and PZS the azimuth $=A$,

then any three of the quantities l, δ, z, h, A being given, the other two may be found by the common rules of Trigonometry.

130. Ex. 1. *To find the azimuth of a heavenly body from its given declination and zenith distance at a given place.*

Here we have PZ, PS, ZS given, and PZS required;

$$\text{therefore } \cos A = \frac{\sin\delta - \sin l \cos z}{\cos l \sin z},$$

or, in a form adapted to logarithmic computation

$$\sin\tfrac{1}{2}A = \sqrt{\frac{\cos\frac{1}{2}.(z+l+\delta).\sin\frac{1}{2}.(z+l-\delta)}{\cos l.\sin z}}.$$

131. Ex. 2. *To find the Sun's azimuth at rising.*

Here $z=90^0$;

$$\text{therefore } \cos A = \frac{\sin\delta}{\cos l}.$$

132. Ex. 3. *To find the Sun's azimuth, when his declination, and the time of day are given.*

Here PZ, PS, ZPS are given to find PZS; therefore ϕ being a subsidiary angle determined by the equation $\tan\phi = \cos h.\cot\delta$,

$$\cot A = \frac{\cos(\phi+l).\cot h}{\sin\phi}.$$

133. If the azimuth, determined by either of the three last articles, be compared with that pointed out by the compass, we obtain the *variation of the compass*, or the deflection of the magnetic meridian from the true meridian.

134. Ex. 4. *To find when the Sun is due east on a given day.*

Here ZS is the prime vertical, and PZS is a right angle;

$$\text{therefore } \cos h = \cot l \,.\, \tan \delta.$$

If z be the corresponding zenith distance,

$$\sin \delta = \sin l \,.\, \cos z,$$

$$\text{therefore } \cos z = \sin \delta \,.\, \operatorname{cosec} l.$$

135. Ex. 5. *To find the Sun's zenith distance at six o'clock.*

$$\text{Here } ZPS = 90^0;$$

$$\text{therefore } \cos z = \sin l \,.\, \sin \delta.$$

136. If a star S (fig. 22.) be nearer to the pole than the observer's zenith Z, its circle of diurnal motion will pass between Z and P; therefore PZS the azimuth, will increase till the vertical circle ZS touches the parallel Ss described by the star, and afterwards diminish. There will in the same manner be another maximum azimuth on the other side the meridian, and at each of these times the star will seem to move in the vertical circle ZS.

Cor. Between the tropics, the Sun's azimuth admits of a maximum twice a day.

137. *To find the hour angle when a star's motion is vertical, the declination of which is greater than the latitude of the observer.*

$$\text{Here } ZSP = 90^0;$$

$$\text{therefore } \cos h = \tan \Delta \,.\, \cot c = \cot \delta \,.\, \tan l.$$

138. *To find the azimuth of a terrestrial object.*

Let S (fig. 23.) be the Sun, S' the object, observe the zenith distances ZS (z), ZS' (z') of the Sun and object, and the arc SS' (d) which their distance subtends at the observer, then

$$\sin \tfrac{1}{2}\,(SZS') = \sqrt{\frac{\sin \tfrac{1}{2}\,.\,(d+z-z')\,.\,\sin \tfrac{1}{2}\,.\,(d+z'-z)}{\sin z\,.\,\sin z'}},$$

and adding SZS', to the Sun's azimuth the azimuth of S' is determined.

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CHAP. VIII.

ON REFRACTION.

139. IN order to explain more simply the phænomena of the heavens, it has been taken for granted in the preceding Chapters,—that the heavenly bodies are seen in the direction of lines drawn to them from the eye of the spectator;—that on account of their great distance, it makes no difference whether they are referred to planes, passing through the place of observation, or to parallel planes drawn through the center of the Earth, or that their altitudes, latitudes, &c. would be the same upon either supposition; and—that the axis round which the diurnal motion is performed, always passes through the same points of the celestial sphere. None of these assumptions are exactly true, therefore the altitudes, latitudes, &c., calculated upon the supposition of their truth, must be increased by certain quantities which are called *corrections.*

The most important corrections are those of *Refraction, Aberration, Parallax, Precession,* and *Nutation,* the first of which will be considered here, and the remainder in the succeeding Chapters.

140. The atmosphere which surrounds the Earth on all sides may be considered made up of spherical laminæ, concentric with the Earth's surface, and diminishing rapidly in density as their distance from the Earth increases. The vertical plane passing through a heavenly body cuts these laminæ perpendicularly; therefore, the rays of light which proceed from the body, and are refracted by the atmosphere according to the common laws of Optics, are bent continually a in vertical plane, so as to describe curves concave towards the surface of the Earth. As the heavenly bodies are seen in the direction in

which the rays of light proceeding from them enter the eye of the observer, or in the direction of a tangent to the extremity of the curve described by the ray, therefore, they appear elevated above their true places. The difference between the true and observed altitudes, or zenith distances, is called the correction of refraction, which it is the design of this Chapter to calculate. The rays which proceed from a star in the zenith suffer no refraction; for inferior altitudes, the refraction increases as the altitude diminishes, and attains its maximum (about $33'$) when the star is seen in the horizon. On account of the refraction taking place wholly in a vertical plane, the azimuth of a heavenly body is not affected by it.

141. By the principles of Optics, if ϕ be the angle of incidence out of one medium into another, r the refraction, and, therefore, $\phi + r$ the angle of emergence, $\sin(\phi + r) = m \cdot \sin\phi$, m being a constant depending on the difference of densities of the media, and determined by experiment. In media of uniform density, the refraction may be calculated immediately from this expression, paying attention to the form of the bounding surface, but the calculation of the refraction for the medium which surrounds the Earth, is rendered more difficult by the density of the air, not only varying at different heights, but also changing at the same place from alterations in the temperature and atmospheric pressure.

The refraction at different temperatures and pressures, may be made to depend on the refraction at the mean heights of the thermometer and barometer.

142. *Given the refraction for a known apparent altitude of a heavenly body, when the barometer is at 29.6 inches, and Fahrenheit's thermometer at 50°, to find the refraction at the same altitude for any other heights of the thermometer and barometer.*

The elastic force of the air at a given temperature varies as its density, and if the density be given increases by .002083 (a) parts of the whole for each degree of temperature above 50°; therefore, if D be the density of the air, when the thermometer is at $50+t$ degrees, the elasticity varies as

$$D.(1+at).$$

The elastic force of the air varies also as the weight of the mercury which it supports in the barometer, which, since mercury expands .000122 (β) parts of its bulk at 50^0 for each additional degree of temperature, varies as $(1-\beta t).b$, b being the height of the barometric column in inches at the time of observation, therefore, $D.(1+at)$ varies as $b.(1-\beta t)$, and

$$D \propto \frac{1-\beta t}{1+at}.b.$$

The refraction at a given apparent altitude of a star depends chiefly upon the density of the air at the place of observation, and is found to vary as it; therefore, if R be the refraction for a given apparent altitude, when the temperature is 50^0, and the barometer 29.6 inches, and r the refraction for the same altitude when the thermometer is 50^0+t^0, and the barometer b inches;

$$r : R :: \frac{1-\beta t}{1+at}.b : 29.6;$$

$$\therefore\ r = \frac{b}{29.6}.\frac{1-\beta t}{1+at}.R.$$

143. The rays which proceed from a star near the zenith, fall nearly perpendicularly on the strata of the atmosphere, and are refracted as they would be by media, bounded by plane surfaces parallel to each other and to the horizon; therefore, the refraction is the same as if the rays of light passed immediately into the last medium, and calling m the ratio of $\sin I : \sin R$ out of a vacuum into the air at the Earth's surface, the equation of Art. 141. becomes

$$\sin(\phi+r) = m\sin\phi,$$

$$\text{or } \sin\phi + \cos\phi r = m\sin\phi, \text{ nearly;}$$

$$\therefore\ r = (m-1).\tan\phi.$$

ϕ is the angle between the last direction of the ray, and a perpendicular to the horizon, or the apparent zenith distance of the star; therefore, for stars near the zenith,

$$\text{the refraction} \propto \tan \text{zenith distance}.$$

144. Cor. It may be proved in the same manner, whatever be the form of the common surface of two contiguous media, that the refraction of a ray which passes from one into the other $= \overline{m-1} \,.\, \tan$ emergence into the denser medium. The coefficient $m-1$ is proved by experiment to vary as the difference of density of the two media; therefore, if q be the density of the air at any distance from the Earth's center, or dq the difference of density of two contiguous strata, the refraction in passing from the one stratum into the other,

$$= \lambda dq \,.\, \tan \text{emergence},$$

λ being some constant. The quantity λ is found by observation to be extremely small, compared with unity, so that its square may be neglected in any calculations without sensible error.

Cassini supposed that the rays of light proceeding from a heavenly body, are refracted as they would be by an homogeneous atmosphere of the density of air at the Earth's surface, and supporting the same mercurial column as that which the atmosphere sustains.

145. *To find the refraction, considering the atmosphere homogeneous.*

Let SDZ (fig. 24.) be the course of a ray of light from a star S, which falls upon the homogeneous spherical atmosphere DO at D, and is refracted in the direction DZ to a spectator on the Earth's surface at Z.

Let DZO the apparent zenith distance of $S = z$, $r =$ the refraction at D, CZ the radius of the Earth $= a$, ZO the height of the homogeneous atmosphere, $= na$, where n is very small, and nearly $= .00125$; therefore $CD = \overline{1+n} \,.\, a$.

$$\text{Then } r = \overline{m-1} \,.\, \tan CDZ \ldots\ldots\ldots (1),$$

$$\text{but } \sin CDZ = \frac{CZ}{CD} \cdot \sin z = \frac{1}{1+n} \cdot \sin z$$

$$= (1-n) \cdot \sin z \text{ nearly,}$$

neglecting n^2;

$$\therefore \cos CDZ = \sqrt{1-(1-n)^2 \cdot \sin^2 z}$$

$$= \sqrt{\cos^2 z + 2n \sin^2 z}$$

$$= \cos z \cdot \sqrt{1+2n \cdot \tan^2 z}$$

$$= \cos z \cdot (1+n \cdot \tan^2 z);$$

$$\therefore r = \overline{m-1} \cdot \overline{1-n} \cdot \frac{\tan z}{1+n \tan^2 z}$$

$$= (m-1) \cdot (1-n) \cdot (\tan z - n \tan^3 z)$$

$$= (m-1) \cdot \{\tan z - n \cdot (\tan z + \tan^3 z)\}$$

$$= (m-1) \cdot (\tan z - n \tan z \cdot \sec^2 z);$$

or expressed in seconds,

$$r = \frac{(m-1)}{\sin 1''} \cdot (\tan z - n \tan z \cdot \sec^2 z).$$

When the barometer is 29.6 inches, and the temperature 50°, $\frac{m-1}{\sin 1''}$ has been found by exact experiments $= 57''.82$; therefore, in general,

$$r = \frac{b}{29.6} \cdot \frac{1-\beta t}{1+at} \cdot 57''.82 \cdot (\tan z - n \tan z \cdot \sec^2 z).$$

The preceding method, which is taken with slight alterations from Dr. Brinkley's Astronomy, is more simple in its demonstration than any other, and gives results exact enough for all observations where the zenith distance does not exceed 80°.

The numerical value of the coefficient given by Dr. Brinkley is

$$\frac{b}{29.6}\cdot\frac{1.0375\,.\,57''.82}{1+.002083\,.(t-32)},$$

where t is the temperature in degrees, by Fahrenheit's thermometer.

The following method of finding the refraction, from Littrow, is independent of any hypothesis on the constitution of the atmosphere, and leads to a differential equation, nearly agreeing with that of Laplace (*Mec. Cel.* Vol. IV. p. 246.), the approximate solution of which, if the coefficients are assumed properly, coincides with the preceding.

146. Let C (fig. 25.) be the center of the Earth, AB its surface, A the place of an observer whose zenith is Z, $Apqr$ the curve described by a ray of light proceeding from a heavenly body, AT a tangent at its extremity, in the direction of which the body will be seen; take qr, a very small arc which may be considered as coinciding with its tangent qT, join Cq, Cr, and draw qs perpendicular to Cr; then, if q = the density of the air at q, $d\rho$ = the refraction at the same point, ρ being the whole refraction from A to q, and CqT the angle of emergence at $q = z$,

$$d\rho=\lambda dq\,.\,\tan z\ldots\ldots\ldots\ldots(1).$$

Let $CA=a$, $Bq=x$, $ACq=v$; $\therefore$ $qCr=dv$, and

$$d\rho=Tqp=Tqc-pqC=qCr+(qrC-pqC)$$

$$=dv+dz,$$

$$\text{but } dv=\frac{qs}{cq}=\frac{dx\,.\tan z}{a+x};$$

$$\therefore\ \lambda dq\tan z=d\rho=dz+dx\,.\frac{\tan z}{a+x};$$

$$\therefore\ \frac{dz}{\tan z}=\lambda dq-\frac{dx}{a+x};$$

$$\therefore \log \sin z = \lambda q - \log (a+x) + \text{corr}.$$

Let δ and Z be the values of q and z at A, where $x = 0$;

$$\therefore \log \sin Z = \lambda \delta - \log a + \text{corr}.$$

$$\therefore \log \frac{\sin z}{\sin Z} = -\lambda . (\delta - q) - \log \frac{a+x}{a}$$

$$= \log . e^{-\lambda(\delta - q)} . \frac{a}{a+x};$$

$$\therefore \sin z = \sin Z . e^{-\lambda(\delta - q)} . \frac{a}{a+x};$$

$$\text{but } d\rho = \lambda dq . \tan z$$

$$= \frac{\lambda \sin z . dq}{\sqrt{1 - \sin^2 z}};$$

therefore, substituting the value of $\sin z$,

$$d\rho = \frac{\lambda . \sin Z . e^{-\lambda . (\delta - q)} . a dq}{\sqrt{(a+x)^2 - \sin^2 Z . e^{-2\lambda . (\delta - q)} . a^2}}.$$

This equation cannot be integrated in general unless some hypothesis be formed as to the relation between the density of the air and its distance from the Earth, but for zenith distances less than 80°, it may be solved in a rapidly converging series, since the greatest value of x, at which light is refracted, is known to be very small compared with a*.

In this case, neglecting the square and higher powers of λ,

$$d\rho = \frac{\lambda . \sin Z \, dq}{\sqrt{1 + \frac{2x}{a} + \frac{x^2}{a^2} - \sin^2 Z}}$$

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* According to Vince, (*Astronomy*, Chap. VII.) the atmosphere does not act upon light at a greater altitude than 77.25 miles.

$$= \frac{\lambda \, . \sin Z dq}{\sqrt{\cos^2 Z + \frac{2x}{a} + \frac{x^2}{a^2}}}$$

$$= \frac{\lambda \tan Z dq}{\sqrt{1 + \frac{1}{\cos^2 Z} \cdot \left(\frac{2x}{a} + \frac{x^2}{a^2}\right)}} \ldots\ldots (2)$$

$$= \lambda \, . \tan Z dq \, . \left\{1 - \frac{x}{a \cos^2 Z}\right\},$$

neglecting the square of $\frac{x}{a}$;

$$\therefore \; \rho = \int \lambda \tan Z dq \, . \left\{1 - \frac{x}{a \cos^2 Z}\right\},$$

the integrals being taken from $q = 0$ to $q = \delta$, and from the greatest value of x to $x = 0$.

The integral of $\lambda \tan Z dq$ between the limits $= \lambda \tan Z \, . \, \delta$,

$$\text{and} \int x dq = qx - \int q dx,$$

of which the part qx vanishes between the limits and the part $\int q dx =$ the atmospheric pressure of the weight of the mercury in the barometer, neglecting the variation of gravity within the limits of x; therefore, if b be the height of the mercury, considering its density unity,

$$\rho = \lambda \delta \tan Z \, . \left\{1 - \frac{b}{\delta a} \cdot \frac{1}{\cos^2 Z}\right\}.$$

The mean value of δ, if the density of mercury is $1 = \frac{1}{10485}$, and the mean values of b and a are about 29.60 inches, and 4000 miles respectively; therefore, the mean value of $\frac{b}{\delta a}$

= .0012517, which may be used without sensible error for the coefficient of $\frac{1}{\cos^2 Z}$; therefore substituting this value

$$\rho = \lambda\delta . \{\tan Z - 0012517 . \tan Z . \sec^2 Z\},$$

or, if D be the density of the air when the thermometer is at 50^0, and the height of the barometer 29.6 inches,

$$\rho = \lambda D . \frac{1-\beta t}{1+at} . \{\tan Z - 0012517 . \tan Z . \sec^2 Z\},$$

where the coefficient λD must be determined by observation.

If the expansion of equation (2), be carried on as far as x^2, the refraction

$$= \lambda D . \frac{1-\beta t}{1+at} . \tan Z \left\{1 - . \frac{0012517}{\cos^2 Z} + 00000139 . \frac{2+\sin^2 Z}{\cos Z^4}\right\}.$$

147. *To determine the coefficient of refraction.*

Let z, z' be the greatest and least observed zenith distance of the same circumpolar star, r, r' the refractions, and c the co-latitude of the place of observation; then, if a be the required coefficient of refraction, which is the quantity designated in the last article by λD, and said, in Art. 145. to equal 57″.82, $r = au$, u representing in this equation a known function depending on the observed zenith distance, and the observed height of the thermometer and barometer, similarly, $r' = au'$, u' being the corresponding function for the zenith distance z';

$$\therefore 2c = z + r + z' + r',$$

$$= z + z' + a . (u + u').$$

If z_1, z_1', u_1, u_1', are corresponding values of z, z', u, u' for another circumpolar star,

$$2c = z_1 + z_1' + a . (u_1 + u_1');$$

$$\therefore z + z' + a . (u + u') = z_1 + z_1' + a . (u_1 + u_1');$$

$$\therefore\ a = \frac{z_1 + z_1' - z - z'}{u + u' - u_1 - u_1'}.$$

The numerical value of a determined by numerous observations, is, as was before observed $=57''.82$.

148. Bradley made use of the solstitial meridian zenith distances of the Sun in determining the coefficient of observation. If s and s' be these distances, r, r' the refractions, ω the obliquity, and l the latitude of the place of observation,

$$l-\omega=s+r,$$

$$l+\omega=s'+r';$$

$$\therefore\ 2l=s+s'+r+r',$$

$$\text{or } 180^0-2c=s+s'+a.(\sigma+\sigma'),$$

σ and σ' being the known functions of s and s',

$$\text{but } 2c=z+z'+a.(u+u');$$

$$\therefore\ 180^0=s+s'+z+z'+a.(\sigma+\sigma'+u+u'),$$

$$\text{and } a=\frac{180^0-s-s'-z-z'}{\sigma+\sigma'+u+u'}.$$

149. Bradley calculated his refractions from the equation $r=57''\tan(z-3r)$, determining r' by the equation $r=57''\tan z$, and then r from the equation $r=57''\tan(z-3r')$. He never gave any demonstration of his formula, and most probably obtained it by conjecture and trial. From Cassini's hypothesis, (Art. 144.) an equation of the same form may be obtained. See *Delambre*, Chap. xiii.

150. If the declination of any one star were known correctly, a table of refraction might be easily formed, by observing its altitude and azimuth at the same instant. From the time elapsed between the observation and the star's transit over the meridian,

the hour angle of the star may be found; and from this hour angle, the star's declination, and the observed azimuth, the true zenith distance of the star, at the instant of observation, may be obtained: having thus found the true and the observed zenith distance, the corresponding refraction is known. In this manner the accuracy of any table of refraction may be ascertained.

The refraction found by the preceding methods cannot be relied on, when the zenith distance is greater than 80^0, beyond this limit it does not depend solely on the state of the atmosphere at the place of observation, but sometimes when the zenith distance is near 90^0, varies at the same place on different days by $3'$ or $4'$, whilst the temperature and barometric pressure remain unaltered.

151. The propositions of the preceding Chapters have been demonstrated without taking into consideration the refractive power of the atmosphere; therefore the altitudes and zenith distances which enter into them, are not the results of direct observation, but are formed by correcting the observed altitudes and zenith distance for the error of refraction, by tables calculated from formulæ similar to those in this Chapter.

Since refraction causes the heavenly bodies to appear elevated above their true places, it accelerates their rising, and retards their setting; but, as the azimuth is not altered by it, the times of transit over the meridian are the same as if there were no refraction.

152. *To find how much the time of a star's rising is altered by refraction.*

Let h be the hour angle when the star really rises,

h' .... appears to rise,

or when the zenith distance $= 90^0 +$ the horizontal refraction (r).

$$\text{Then } \cos h = \frac{\cos 90^0 - \sin l \,.\, \sin \delta}{\cos l \,.\, \cos \delta} = - \frac{\sin l \,.\, \sin \delta}{\cos l \,.\, \cos \delta};$$

$$\cos h' = \frac{\cos(90+r) - \sin l . \sin \delta}{\cos l . \cos \delta} = - \frac{\sin r + \sin l . \sin \delta}{\cos l . \cos \delta};$$

$$\text{therefore } \cos h - \cos h' = \frac{\sin r}{\cos l . \cos \delta},$$

$$\text{or } 2 \sin \tfrac{1}{2}(h' - h) . \sin \tfrac{1}{2}(h' + h) = \frac{\sin r}{\cos l . \cos \delta},$$

or, since $h' - h$ and r are very small,

$$(h' - h) . \sin h = \frac{r}{\cos l . \cos \delta};$$

$$\therefore h' - h = \frac{r}{\cos l . \cos \delta . \sin h},$$

$$\text{but } \cos l . \cos \delta . \cos h = - \sin l . \sin \delta;$$

$$\text{therefore } \cos^2 l . \cos^2 \delta . \sin^2 h = \cos^2 l . \cos^2 \delta - \sin^2 l . \sin^2 \delta$$
$$= \cos(l + \delta) . \cos(l - \delta);$$

$$\therefore h' - h = \frac{r}{\sqrt{\cos(l + \delta) . \cos(l - \delta)}}.$$

from which the difference of times $\frac{h' - h}{15^0}$ is known.

153. Cor. Let T and T' be the times in which a star in the equator is above and below the horizon, then $T - T'$ is the error in the times of rising and setting caused by refraction, for without refraction, T would equal T';

$$\therefore T - T' = \frac{1}{15^0} . \frac{2r}{\cos l}, \text{ since } \delta = 0;$$

$$\therefore r = \tfrac{1}{2} . 15^0 . (T - T') . \cos l.$$

The mean value of r, found in this way, is 33′.

154. Cor. 2. On account of the smallness of r, the equation $h' - h = \frac{r}{\sqrt{\cos(l+\delta) . \cos(l - \delta)}}$ gives the acceleration in

the time of rising with sufficient accuracy, h and h' may however be determined exactly from the equations

$$\cos h' = -\tan l \,.\, \tan \delta,$$

$$\cos h' = -\sin r \,.\, \sec l \,.\, \sec \delta - \tan l \,.\, \tan \delta,$$

from whence $h' - h$ may be found.

From the effects of refraction increasing with the zenith distance, the lower limb of the Sun or Moon is more elevated than the upper one, in consequence of which the disks of these bodies appear oval instead of round. This effect is most sensible in the horizon, where the variation of refraction is greatest.

155. *To find the alteration produced by refraction on any diameter of the Moon.*

The differences of refraction for the small extent of the lunar disk may be considered proportional to the differences of altitude; therefore, if the horizontal diameter of the Moon be taken for the axis of x, the ordinates perpendicular to it will be diminished in proportion to their length, and the circular disk will become an ellipse.

Let Δ be the Moon's apparent horizontal diameter; and therefore $\Delta - r$ the vertical diameter, r being the difference of refraction for a difference of altitude Δ, then the equation of the ellipse is

$$y^2 = \frac{(\Delta - r)^2}{\Delta^2} \,.\, \left\{\tfrac{1}{4} \,.\, \Delta^2 - x^2\right\}.$$

If $2R$ be any diameter inclined at an angle a to the horizontal diameter, $y = R \sin a$, and $x = R \cos a$;

$$\therefore R^2 \sin^2 a = \tfrac{1}{4} \,.\, (\Delta - r)^2 - \frac{(\Delta - r)^2}{\Delta^2} \,.\, R^2 \cos^2 a;$$

$$\therefore 2R = \frac{(\Delta - r) \,.\, \Delta}{\sqrt{\Delta^2 \sin^2 a + (\Delta - r)^2 \cos^2 a}}$$

$$= \frac{\Delta^2 - \Delta r}{\sqrt{\Delta^2 - 2\Delta r \cos^2 a}}, \text{ nearly,}$$

$$= \Delta . \left(1 - \frac{r}{\Delta}\right) . \left\{1 + \frac{r}{\Delta} \cos^2 a\right\}$$

$$= \Delta . \left\{1 - \frac{r}{\Delta} \sin^2 a\right\}.$$

But if δ (fig. 26.) be the true horizontal diameter LM of the Moon, the extremities of which L and M are raised by refraction to l and m in the verticals ZL, ZM, $lm = \Delta$. Draw ZO bisecting LM and lm, and let $Zo = z$, then $Oo = r$;

$$\therefore \; \delta : \Delta :: \sin(z + r) : \sin z$$

$$:: 1 + r \cot z : 1, \text{ nearly,}$$

but since $r \propto \tan z$ nearly, $r \cot z$ is constant, let it equal c;

$$\therefore \Delta = \frac{\delta}{1 + c};$$

$$\therefore \; 2R = \frac{\delta}{1 + c} . \left\{1 - \frac{r}{\delta} . (1 + c) . \sin^2 a\right\}$$

$$= \delta . (1 - c) . \left\{1 - \frac{r}{\delta} \sin^2 a\right\},$$

neglecting c^2 and rc.

156. After sun-set, or before sun-rise, many of the Sun's rays enter the atmosphere, and are reflected by it to the Earth in an irregular manner, which causes the pale light called *Twilight*. The effect continues as long as the Sun is not more than a certain distance beneath the horizon, which varies in different latitudes from 16° to 20°, and may be taken at the mean value 18°, as the beginning and end of twilight cannot be observed with much accuracy.

157. The duration of twilight may be found, as the acceleration of a star's rising by refraction was in Art. 152. If h

and h' be the hour angles corresponding to the beginning and end of twilight, and l and δ as before,

$$\cos h = -\tan l \,.\, \tan \delta,$$

$$\cos h' = -\sin 18^0 \,.\, \sec l \,.\, \sec \delta - \tan l \,.\, \tan \delta.$$

From which h and h', and the difference $h'-h$, may be deduced.

The greatest depression of the Sun below the horizon = the distance of the equator from the horizon — the Sun's declination

$$= 90^0 - l - \delta;$$

therefore twilight will continue all night, if

$$90^0 - l - \delta \text{ is less than } 18^\circ,$$

$$\text{or } l + \delta \text{ greater than } 72^0.$$

If $l + \delta = 72^0$, the evening twilight will end, when the morning begins.

158. *To find the time of year when the twilight is shortest.*

Let Z (fig. 27.) be the zenith, P the pole, ZQ a parallel to the equator, S the Sun at the commencement of twilight; therefore $ZS = 108$. Let the great circle which has S for its pole, cut ZQ in Z'; then, ZPS is the hour angle when twilight begins, and since PZ' is the co-latitude, PS the north polar distance of the Sun, and $ZS = 90^0$; therefore $Z'PS =$ the hour angle at sun-rise; therefore the difference of these angles ZPZ' measures the duration of twilight, and is least when ZZ' is least; but ZZ' is never less than $SZ - SZ'$ or 18^0, and is least when it exactly equals $SZ - SZ'$, in which case Z' coincides with Q, and $ZQ = SZ - SZ' = 18^0$. Draw PO perpendicular to SZ, and bisecting ZQ and ZPQ, and let l be the latitude of the place, δ the declination of the Sun, and $ZPQ = 2h$.

Then, by Napier's Rules,

$$\cos ZP = \cos PO \,.\, \cos ZO, \quad \text{or} \quad \sin l = \cos PO \,.\, \cos 9^0,$$

$$\cos PS = \cos PO \,.\, \cos SO, \quad \text{or} \quad \sin \delta = -\cos PO \,.\, \sin 9^0;$$

$$\therefore \frac{\sin \delta}{\sin l} = - \tan 9^0,$$

$$\text{and } \sin \delta = - \tan 9^0 . \sin l \ldots . (1).$$

$$\text{Again, } \sin ZO = \sin ZP . \sin ZPO,$$

$$\text{or } \sin 9^0 = \cos l . \sin h;$$

$$\therefore \sin h = \sin 9^0 . \sec l \ldots\ldots (2).$$

The equation (1) gives the Sun's declination, or the time when the twilight is shortest, and equation (2) gives the duration of it.

◆

CHAP. IX.

ON PARALLAX.

159. Let O (fig. 28.) be a spectator on the Earth's surface, Z his zenith, C the center of the Earth; then, a heavenly body p seen from O has a zenith distance ZOp, and from C, a zenith distance ZCp, less than ZOp by the angle at p. This angle, which the Earth's radius subtends at p, is called the parallax of p, and must be added to the true zenith distance seen from the center of the Earth, in order to give the apparent distance as seen from O.

Cor. Hence it appears that the effect of parallax is to depress bodies in a vertical plane, and since this is contrary to the effect of refraction, it hastens the setting, delays the rising, and diminishes the effects of refraction on the diameters of the heavenly bodies.

160. *To find the parallax of a known body.*

Let CO (fig. 28.) the distance of the observer from the Earth's center $= r$,

$$Cp = R,\ ZCp = z,\ OpC = p, \text{ and } ZOp = z + p = z';$$

$$\therefore \sin(z+p) : \sin p :: R : r;$$

$$\therefore \sin p = \frac{r}{R} . \sin(z+p) \ldots\ldots\ldots (1)$$

$$= \frac{r}{R} . (\sin z . \cos p + \cos z . \sin p);$$

therefore, dividing by $\cos p$,

$$\tan p = \frac{r}{R} \cdot \frac{\sin z}{1 - \frac{r}{R} \cdot \cos z} \ldots\ldots (2).$$

161. Cor. 1. The parallax is generally so small that no sensible error is introduced by making $\sin p = p$; therefore, from (1),

$$p = \frac{r}{R} \cdot \sin z' \propto \sin z' \propto \sin z, \text{ nearly.}$$

162. Cor. 2. Since $\sin p = \frac{r}{R} \cdot \sin z'$; therefore, p diminishes as R increases, and although p can never be really 0, yet if R exceed $\frac{r}{\sin 1''}$, p must be less than 1″, and would be too small to admit of detection by our instruments. The fixed stars have no sensible parallax, and in this the effect of parallax differs from that of refraction, that the latter is independent of the distance of the body observed, whilst the former diminishes as the distance increases, and at length becomes insensible.

163. Cor. 3. When the body is in the horizon,

$$z + p = 90^\circ;$$

if therefore P be the value of p in this case, $\sin P = \frac{r}{R}$;

$$\therefore \sin p = \sin P \cdot \sin z',$$

$$\text{or } p = P \cdot \sin z' \ldots\ldots\ldots\ldots (4).$$

164. Cor. 4. If P' be the horizontal parallax at the equator, and A the radius of the equator,

$$P' = \frac{A}{R};$$

$$\therefore P = \frac{r}{R} \,.\, P'.$$

165. Cor. 5. Since $\sin p = \sin P \,.\, \sin (z+p)$;

$$\therefore e^{p\sqrt{-1}} - e^{-p\sqrt{-1}} = \sin P . e^{\overline{z+p}\sqrt{-1}} - \sin P . e^{-\overline{z+p}\sqrt{-1}};$$

$$\therefore e^{2p\sqrt{-1}} - 1 = \sin P . e^{z\sqrt{-1}} . e^{2p\sqrt{-1}} - \sin P . e^{-z\sqrt{-1}};$$

$$\therefore e^{2p\sqrt{-1}} = \frac{1 - \sin P . e^{-z\sqrt{-1}}}{1 - \sin P \,.\, e^{z\sqrt{-1}}};$$

$$\therefore 2p\sqrt{-1} = \log (1 - \sin P . e^{-z\sqrt{-1}}) - \log (1 - \sin P . e^{z\sqrt{-1}})$$

$$= \sin P . (e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}) + \tfrac{1}{2} . \sin^2 P . (e^{2z\sqrt{-1}} - e^{-2z\sqrt{-1}}) + \&c.$$

$$\therefore p = \sin P \,.\, \sin z + \tfrac{1}{2} \sin^2 P \,.\, \sin 2z + \&c.$$

166. *To find the effects of parallax on the hour angle and declination of a known body.*

Let Z (fig. 29.) be the zenith of any place, P the pole of the equator, S the true place of a heavenly body, depressed by parallax to S' in the vertical ZS, draw SO perpendicular to PS', and let α (SPS') be the parallax of the hour angle ZPS (h), and β ($S'O$) the parallax in declination.

$$\text{Then } \alpha = \frac{SO}{\sin PS} = \frac{SS' . \sin ZS'P}{\sin PS}$$

$$= P . \frac{\sin ZS' . \sin ZS'P}{\sin PS}$$

$$= P . \frac{\sin ZP \,.\, \sin ZPS'}{\sin PS}$$

$$= P . \frac{\cos l \,.\, \sin (h+\alpha)}{\cos \delta};$$

l being the latitude of Z.

$$\beta = S'O = SS' . \cos SS'O$$

$$= P . \sin ZS' . \frac{\cos ZP - \cos ZS' . \cos PS'}{\sin ZS' . \sin PS'}$$

$$= \frac{P}{\sin PS'}\{\cos ZP - \cos PS' . (\sin ZP . \sin PS' . \cos ZPS' + \cos ZP . \cos PS')\}$$

$$= \frac{P}{\sin PS'}\{\cos ZP . \sin^2 PS' - \cos PS' . \sin ZP . \sin PS' . \cos ZPS'\}$$

$$= P\{\cos ZP . \sin PS' - \cos PS' . \sin ZP . \cos ZPS'\}$$

$$= P\{\sin l \cos \delta - \cos l \sin \delta . \cos (h + a)\}.$$

Let a' be determined by the equation

$$a' = P . \frac{\cos l . \sin h}{\cos \delta};$$

then $a = P . \dfrac{\cos l . \sin (h + a')}{\cos \delta}$,

$$\beta = P . \{\sin l \cos \delta - \cos l . \sin \delta . \cos (h + a')\}.$$

167. Cor. 1. The error in right ascension of S, caused by parallax, is the same as that of the hour angle, and $ZPS =$ the right ascension of the mid-heaven (A) — the right ascension of S (a); therefore, the parallax in right ascension

$$= P . \frac{\cos l . \sin (A - a)}{\cos \delta}.$$

168. Cor. 2. If P be the pole of the ecliptic, $S'O$ is the parallax in latitude, and SPS' the parallax in longitude of the star S; therefore, the parallax in longitude

$$= P . \frac{\sin ZP . \sin ZPS}{\sin PS};$$

and the parallax in latitude

$$= P . \{\cos ZP . \sin PS - \cos PS . \sin ZP . \cos ZPS\}.$$

In this case ZP is the height of the nonagesimal (n), and ZPS or ZPS' the longitude of the nonagesimal (L) – the longitude of S (l); therefore, if λ be the body's latitude, the parallax in longitude

$$= P . \frac{\sin n . \sin (L - l)}{\cos \lambda};$$

the parallax in longitude

$$= P \left\{ \cos n \cos \lambda - \sin n . \sin \lambda . \cos (L - l) \right\}.$$

169. *To find the augmentation by parallax of the Moon's diameter.*

Let D be the apparent diameter of the Moon p (fig. 28.), seen from the center of the Earth, and D' the diameter seen from O, then since the body is the same,

$$D' : D :: Cp : Op :: \sin z' : \sin (z' - p);$$

$$\therefore \; D' = D . \frac{\sin z'}{\sin (z' - p)};$$

$$\therefore \; D' - D = D . \frac{\sin z' - \sin (z' - p)}{\sin (z' - p)};$$

$$= 2 D . \frac{\sin \frac{1}{2} . p . \cos (z' - \frac{1}{2} p)}{\sin (z' - p)}.$$

Cor. If Δ be the value of D' when the Moon is in the horizon, in which case $z' = 90^0$, and p = the horizontal parallax P,

$$\Delta - D = 2 D . \frac{\sin^2 \frac{1}{2} P}{\cos P} = D . \frac{1 - \cos P}{\cos P};$$

$$\therefore \; D = \Delta \cos P.$$

The same formulæ may be applied to the Sun or any of the planets, but these bodies are too distant for parallax to produce any sensible effect on their apparent diameters.

170. *To find the true latitudes and longitudes of a heavenly body, from the latitudes and longitudes affected by parallax.*

Let l, λ, r, be the longitude, latitude, and distance of a heavenly body respectively, as seen from the Earth's center, l', λ', r' the corresponding quantities from the surface, Λ and L the latitude and longitude of the observer's zenith, and R his distance from the Earth's center, which $= r' \sin P$. Let the planet's place be referred to rectangular co-ordinates xyz, the center of the Earth being the origin, and the axis of x passing through the vernal equinox, and that of z through the pole of the ecliptic. Similarly, let X, Y, Z, be the co-ordinates of the observer's place, and let x', y', z' be the co-ordinates of the planet, referred to parallel axes passing through the place of observation. Then

$$x = r \cos\lambda . \cos l, \quad y = r \cos\lambda . \sin l, \quad z = r \sin\lambda \ldots\ldots (1),$$

$$x' = r' \cos\lambda' \cos l', \quad y' = r' \cos\lambda' \sin l', \quad z' = r' \sin\lambda' \ldots . (2),$$

$$X = R\cos\Lambda . \cos L, \quad Y = R\cos\Lambda . \sin L, \quad Z = R\sin\Lambda \ldots . (3),$$

$$= r'\sin P\cos\Lambda . \cos L, \quad = r'\sin P . \cos\Lambda . \sin L, \quad = r'\sin P . \sin\Lambda;$$

$$\therefore\ x = x' - X = r' . \{\cos\lambda' \cos l' - \sin P . \cos\Lambda . \cos L\},$$

$$y = y' - Y = r' . \{\cos\lambda' \sin l' - \sin P . \cos\Lambda . \sin L\},$$

$$z = z' - Z = r' . \{\sin\lambda' - \sin P . \sin\Lambda\}.$$

But, from the equations (1),

$$\tan l = \frac{y}{x}, \quad \tan\lambda = \frac{z}{x} . \cos l;$$

$$\therefore\ \tan l = \frac{\cos\lambda' \sin l' - \sin P . \cos\Lambda . \sin L}{\cos\lambda' \cos l' - \sin P . \cos\Lambda . \sin L},$$

$$\tan\lambda = \frac{\{\sin\lambda' - \sin P . \sin\Lambda\} . \cos l}{\cos\lambda' \cos l' - \sin P . \cos\Lambda . \cos L},$$

From the first of which l may be found, and then λ from the second, and these being known, r may be obtained by the equation

$$r = \frac{x}{\cos\lambda \,.\, \cos l} = r' . \frac{\{\cos\lambda' . \cos l' - \sin P \,.\, \cos\Lambda \,.\, \cos L\}}{\cos\lambda \,.\, \cos l}.$$

171. If D be the apparent diameter of the body seen from the center of the Earth, D' the diameter at the place of observation,

$$D = \frac{r'}{r} . D'$$

$$= \frac{D' . \cos\lambda \,.\, \cos l}{\cos\lambda' . \cos l' - \sin P \,.\, \cos\Lambda \,.\, \cos L}.$$

Λ, the latitude of the observer's zenith, is the same as the zenith distance of the nonagesimal, and L, the longitude of the zenith, the same as the longitude of the nonagesimal.

172. In the same manner, if a and δ be the right ascension and declination of the planet as seen from the center of the Earth, a' and δ' at the place of observation, A the right ascension of the zenith, or of the mid-heaven, and L the declination of the place of observation as seen from the center of the Earth, which is equal to the latitude of the place of observation;

$$\tan a = \frac{\cos\delta' . \sin a' - \sin P \,.\, \sin A \,.\, \cos L}{\cos\delta' \cos a' - \sin P \,.\, \cos A \,.\, \cos L},$$

$$\tan\delta = \frac{\{\sin\delta' - \sin P \,.\, \sin L\} \cos a}{\cos\delta' . \cos a' - \sin P \,.\, \cos L \,.\, \cos A}.$$

173. *To measure the parallax of a heavenly body, by observations made in the plane of the meridian.*

Let ZAP (z) (fig. 30.), $ZA'P$ (z') be two zenith distances of a body P, observed from two known places A and A', on the same meridian; take C the center of the Earth, draw the lines

CP, CAa, $CA'a'$, and let $CA = r$, $CA' = r'$, $CP = R$, $l =$ the latitude of A, $l' =$ the latitude of A'; then, if ϕ, ϕ' be the known angles between the normals at A and A', and the lines CA, CA' (Art. 65.), $PAa = z - \phi$, $PA'a' = z' - \phi'$; therefore if p, p' be the parallaxes at A and A', we have

$$\text{from the triangle } CAp \ldots\ldots \; p = \frac{r}{R} \cdot \sin(z - \phi),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \; CA'p \ldots\ldots \; p' = \frac{r'}{R} \cdot \sin(z' - \phi');$$

$$\therefore p + p' = \frac{r \cdot \sin(z - \phi) + r' \cdot \sin(z' - \phi')}{R}.$$

$$\begin{aligned} \text{But } p + p' &= APA' \\ &= PAa + P'Aa' - ACA' \\ &= z - \phi + z' - \phi' - (l - \phi + l' - \phi') \\ &= z + z' - l - l'; \end{aligned}$$

$$\therefore R = \frac{r \cdot \sin(z - \phi) + r' \cdot \sin(z' - \phi')}{z + z' - l - l'};$$

therefore if P' be the equatoreal parallax, and A the radius of the Earth's equator,

$$P' = A \cdot \frac{z + z' - l - l'}{r \sin(z - \phi) + r' \sin(z' - \phi')},$$

A and A' are supposed on different sides of the equator; if they are on the same side, l' is negative.

174. Cor. If p and p' are too large to be considered equal to their sines, we have

$$\sin p = \frac{r}{R} \cdot \sin(z - \phi) \ldots\ldots\ldots (1),$$

$$\sin p' = \frac{r'}{R} \cdot \sin(z' - \phi'),$$

and calling $z+z'-l-l'=m$, or making $p'=m-p$,

$$\sin(m-p) = \frac{r'}{R}\,.\,\sin(z'-\phi');$$

therefore $\sin m \cos p - \cos m \sin p = \frac{r'}{R} \sin(z'-\phi')$,

and dividing this by equation (1);

$$\sin m\,.\,\cot p - \cos m = \frac{r'}{r}\,.\,\frac{\sin(z'-\phi')}{\sin(z-\phi)};$$

$$\text{therefore } \tan p = \frac{r\,.\,\sin(z-\phi)\,.\,\sin m}{r\,.\,\sin(z'-\phi') + r\,.\,\cos m\,.\,\sin(z-\phi)}.$$

Whence p may be found, and hence

$$R = r\,.\,\frac{\sin(z-\phi)}{\sin p}, \text{ and therefore } P' = \frac{A}{R}.$$

175. By this method the parallax of a heavenly body, the declination of which is unchangeable, may be found if the observers are on different meridians; but if the declination, and, consequently, the meridian zenith distance changes, as is the case with all the bodies which have a sensible parallax, the meridian zenith distance at A' must be observed for several successive days, from which, by the method of interpolations, we may find the meridian zenith distance as seen from a place in the same latitude as A' at the time when the body was on A's meridian; that is, the meridian distance which it would have at A', if it were to move from the meridian of A to that of A', without changing its declination.

176. *To find the Moon's parallax, by observations made out of the plane of the meridian.*

Let l be the latitude of the observer, δ the declination of the Moon, h the hour angle calculated from an observed zenith distance, and which is therefore affected with the error of parallax $a=$

$$P\,.\,\frac{\cos l\,.\,\sin h}{\cos \delta}.\quad \text{(Art. 166).}$$

Let a', h' be the values of a and h, corresponding to another observation, then, as before,

$$a' = P.\frac{\cos l\,.\,\sin h'}{\cos \delta},$$

neglecting the small change of declination between the observations;

$$\therefore\ a - a' = P.\cos l\,.\,\frac{\sin h - \sin h'}{\cos \delta}$$

$$= 2P.\frac{\cos l}{\cos \delta}.\sin \tfrac{1}{2}(h - h').\cos \tfrac{1}{2}.(h + h'),$$

h and h' are known from the observed zenith distances; and therefore $h - h'$ the apparent decrement of the hour angle between the observations; but by the tables of the Moon's motion, or from observing the increase of right ascension on successive days, the true value of $h - h'$ in the known interval between the observations may be found, whnece the difference of these quantities, or that which arises from parallax, $a - a'$ is determined; therefore we know

$$P = \tfrac{1}{2}\,.\,\frac{(a - a').\cos \delta}{\sin \tfrac{1}{2}(h - h').\cos \tfrac{1}{2}.(h' + h).\cos l}.$$

Cor. If the observations are made on different sides of the meridian

$$P = \tfrac{1}{2}.\frac{(a + a').\cos \delta}{\sin \tfrac{1}{2}.(h + h').\cos \tfrac{1}{2}.(h' - h).\cos l}.$$

177. The quantity h must be calculated from the observed zenith distance corrected for refractions. This correction may be avoided by observing at the same time a star nearly in contact with the Moon, then, since the effect of refraction is nearly the same on both, the difference of the hour angles calculated from the observed zenith distances, is equal to the difference of the star's true right ascension, and the Moon's apparent right ascen-

sion, and as the star's right ascension is known, the apparent right ascension of the Moon may be found. Having found, in the same manner, the apparent right ascension of the Moon at the second observation, the difference of these, or the apparent motion in right ascension of the Moon may be obtained; therefore, as before, taking the difference of the Moon's true and apparent motion in right ascension, we obtain $a - a'$.

By the two preceding methods, the Moon's parallax may be determined with considerable accuracy; they may also be applied to the nearer planets, but the Sun's parallax, which is one of the most important elements of Astronomy, is too small to be discovered in this way, as a slight error of observation may entirely vitiate the result. An accurate mode of determining the solar parallax, is described in a subsequent Chapter.

178. The angle subtended at a body by the Earth's radius, is called the *diurnal parallax*, to distinguish it from the angle subtended by the radius of the Earth's orbit at a fixed star, which is called the star's *annual parallax*. It is agreed, that the former of these does not admit of observation for any of the fixed stars, but as the radius of the Earth's orbit is more than 20000 times greater than the radius of the Earth, Astronomers have sought with great care to find by observations, if any of the stars have a sensible annual parallax. They are as yet undecided about its existence, which certainly does not, in any case, exceed $5''$, and is, probably, much less.

179. *Given the greatest annual parallax of a fixed star, to find the parallax in latitude and longitude.*

Let P (fig. 29.) be the pole of the ecliptic, the Sun being considered the center of the sphere, Z the point in which the line passing through the Sun and Earth, intersects the surface of the sphere, S a fixed star; then $PZ = 90^0$, and $ZPS =$ the longitude of the Earth seen from the Sun $(\oplus)$—the longitude of $S\,(l), = \oplus - l$. The parallax will take place in the great circle, ZS passing through the Earth and star for the same reason, that the diurnal parallax takes place in a vertical circle; if, therefore,

ϕ be the greatest parallax, when $ZS = 90^0$, and we proceed as in Art. 166,

the parallax in *longitude*

$$= \phi \cdot \frac{\sin ZP \cdot \sin ZPS}{\sin PS} = \phi \cdot \frac{\sin (\oplus - l)}{\cos \lambda} \dots \dots (1),$$

the parallax in *latitude*

$$= \phi \cdot (\sin ZP \cdot \sin PS - \cos PS \cdot \sin ZP \cdot \cos ZPS)$$

$$= -\phi \cdot \sin \lambda \cdot \cos (\oplus - l) \dots\dots\dots\dots\dots (2).$$

The parallax in latitude can never exceed ϕ; and is, therefore, not so proper for deducing ϕ from observation, as the parallax in longitude.

In the figure from which the parallaxes in the preceding Articles are derived, the latitudes, longitudes, &c. are diminished by parallax; therefore, the values deduced must be subtracted from the true latitudes, longitudes, &c. to get the apparent.

180. *To find, by observation, the annual parallax of a given fixed star, if it has any.*

Let the star's longitude be found, when $\oplus - l = 90^0$, then if l be the longitude seen from the Sun, the apparent longitude

$$= l - \phi \cdot \sec \lambda.$$

At the end of half a year, $\oplus - l = 270^0$, and the apparent longitude

$$= l + \phi \cdot \sec \lambda;$$

therefore, if Δ be the observed difference of the star's longitude at these times,

$$\Delta = 2\phi \cdot \sec \lambda;$$

$$\therefore \phi = \tfrac{1}{2} \cdot \Delta \cos \lambda.$$

181. All the fixed stars have slight *apparent* changes of place, arising from causes which are well known, and which will be explained in the Chapters on Precession and Aberration. Besides these, many of the stars have other changes of position,

which are generally believed to be proper motions, though some, who considered that they were not independent of each other, have thought that they were only apparent, and caused by the motion of our system about some distant point, perhaps the center of gravity of the universe. The effects of such a motion of the Earth upon the apparent places of the stars, will evidently be similar to that produced by parallax, and may be calculated by similar formulæ, but as the Earth's radius was considered evanescent in the investigations of the annual parallax, so in these motions, which, (if the hypothesis of their cause be correct) arise from a *secular* parallax, the whole of the solar system must be considered as occupying a mere point.

If AA' (fig. 31.) be the arc described in a century by our solar system, an immoveable star C will be seen in the directions AC, $A'C$, when the Earth is at A and A'; therefore, considering AA' a straight line, on account of the immense radius of the orbit of which it forms a part, and making ACA' the secular parallax $=\pi$, $CAA'=m$, $AA'=r$, and $AC=\rho$,

$$\sin\pi = \frac{r}{\rho}\,.\sin m.$$

In this equation r and ρ are unknown, and will probably remain so; therefore the absolute secular parallax cannot be discovered. We have it, however, in our power to determine by observation the direction of the motion of the solar system, if any such motion exists.

182. Let P (fig. 29.) be the pole of the equator, Z that part of the heavens towards which our system is supposed to move, S a fixed star, which, in consequence of the Earth's motion from the center of the sphere towards Z, appears to describe the small arc SS' in ZS produced; then, if x be the right ascension, and y the declination of the point Z, a and δ the right ascension and declination of S, m and n the apparent motions of S in right ascension and declination, and ϕ the motion which a star 90^0 from Z would appear to have; we have, as in Art. 166.

$$m = \phi . \frac{\sin ZP . \sin ZPS}{\sin PS}$$

$$= \phi . \frac{\cos y . \sin (x - a)}{\cos \delta} \text{ .... } (1),$$

$$n = \phi . \{\cos ZP . \sin PS - \cos PS . \sin ZP . \cos ZPS\}$$

$$= \phi \{\sin y . \cos \delta - \sin \delta . \cos y . \cos (x - a)\} \text{ ....} (2).$$

Eliminating ϕ between (1) and (2), and substituting for $\sin (x-a)$, $\cos (x-a)$, their expansions

$$m . \{\sin y \cos \delta - \sin \delta . \cos y . (\cos x . \cos a + \sin x . \sin a)\}$$

$$= n \cos y . \frac{(\sin x \cos a - \cos x \sin a)}{\cos \delta};$$

therefore, multiplying by $\cos \delta$, and dividing by $\sin y$,

$$m \{\cos^2 \delta - \sin \delta \cos \delta \cos a . \cot y . \cos x$$
$$- \sin \delta . \cos \delta . \sin a \cot y \sin x\};$$
$$= n \cos a \cot y \sin x - n \sin a \cot y \cos x;$$

$$\therefore m \cos^2 \delta = (m \sin \delta \cos \delta \cos a - n \sin a) \cot y . \cos x$$
$$+ (m \sin \delta \cos \delta \sin a + n \cos a) . \cot y . \sin x,$$

or calling the known coefficients p and q,

$$m \cos^2 \delta = p . \cot y \cos x + q \cot y \sin x.$$

Another simple equation may be formed in the same manner from the apparent motions of another star, and from these two equations the values of $\cot y . \cos x$, $\cot y . \sin x$, and, therefore, of y and x may be determined.

183. When the numerical values deduced from observation are employed for determining y and x, the values which are deduced from the observations of different stars, differ considerably from each other, whence it may be concluded, that the apparent motion of the stars does not arise solely from the motion of the solar system.

CHAP. X.

ON PRECESSION AND NUTATION.

184. THE comparison of ancient and modern observations shews, that the longitude of every fixed star has an annual increase of about 50″, whilst the latitude remains nearly unchanged. This increase which arises from the regression of the first point of Aries, from which the longitudes are reckoned, and not from a real motion of the stars, is called the *Precession of the Equinoxes.*

The same observations point out a cotemporary decrement of the obliquity, which is called the *Secular Diminution of the Obliquity*, and amounts to about half a second.

Both these uniform changes are subject to variations, which recur after a period of 18 years, and are called *Nutations.*

Precession and Nutation arise principally from the attractive forces of the Sun and Moon acting on the protuberant matter of the Earth's equator. These forces may be resolved into two, one in the plane of the equator which cannot change the position of that plane, and the other perpendicular to it, which, if the Earth had no rotation, would diminish the inclination of the planes of the ecliptic and equator, without altering the line of their intersection, but, in consequence of the Earth's rotation, these effects are reversed, and the inclination remains unaltered, whilst the line of equinoxes moves backwards on the ecliptic through 50″.34 annually.

The plane in which the Moon moves is not fixed, as will be shewn hereafter, but changes its position in a period of 18 years, during which time the intersection of the Moon's orbit, and the ecliptic, or the *Line of Nodes*, as it is called, recedes on the ecliptic through 360^0; this causes the effect of the Moon's action to vary, the mean effect being involved in the precession $50''.34$, and the variable part being the *Lunar Nutation*. The Node in which the Moon is, when it passes from the south to the north side of the ecliptic, is called the Moon's *Ascending Node*.

The planets are too distant to produce any sensible effect on the protuberant matter of the Earth, but by their mutual attractions, they produce a slight change in the position of the ecliptic, causing the secular diminution of the obliquity, and a motion forwards of $16''$ in the equinoxial points, which being subtracted from $50''.34$, leaves $50''.18$ for the whole annual regression of the equinoxes. These effects of the planetary attractions are independent of the figure of the Earth, but as they change the plane of the Sun's motion, they must cause a corresponding change in the Sun's action on the protuberant matter of the Earth, similar to the lunar nutation, but much less in quantity, and immensely greater in its period.

185. Let ΥC (fig. 32.) be the ecliptic at a determined epoch, $\Upsilon'' C'$ the ecliptic as changed by the action of the planets at any other time, ΥQ, $\Upsilon' Q'$ the corresponding equators. Then $\Upsilon\Upsilon'$ is the precession caused by the action of the Sun and Moon on the Earth; $N\Upsilon'' - N\Upsilon$ the true precession.

Let $\Upsilon\Upsilon' = \psi$, $N\Upsilon'' - N\Upsilon = \phi$, $N\Upsilon = \pi$, $\Upsilon' = \omega'$, $\Upsilon'' = \omega''$, $\Upsilon N \Upsilon'' = p$. Then $N\Upsilon'' = \pi + \phi$. The quantities ψ, ϕ, ω', ω'' are found by Physical Astronomy, and the comparison of distant observations, and π and p are deduced from them by means of Spherical Trigonometry. If the fixed epoch be 1750, their values for the year $1750 + t$, where t is expressed in years, or parts of years, are

$$\psi = 50''.340499\,t - 0''.00012179\,t^2,$$

$$\phi = 50''.176068\,t + 0''.00012214\,t^2,$$

$$\omega' = 23^0\ 28'\ 18'' + 0''.00000984\,t^2,$$

$$\omega'' = 23^0\ 28'\ 18'' - 0''.48368\,t - 0.0000027\,t^2,$$

$$\pi = 171^0\ 36'\ 10'' - 5''.18\,t,$$

$$p = 0''.48892\,t - 0''.0000030719\,t^2.$$

186. *Given the latitude and longitude of a fixed star at one epoch, to find them at any other.*

Retaining the same construction and notation, let S (fig. 32.) be a fixed star, $SC\,(\lambda_1)$, its latitude at the fixed epoch 1750, $SC'\,(\lambda)$ its latitude at any time $1750+t$, l_1 and l the corresponding longitudes.

$$\begin{aligned}\text{Then } l &= \Upsilon''N + NO + C'O \\ &= \Upsilon''N + NC + C'O \\ &= \Upsilon''N + l_1 - N\Upsilon + C'O \\ &= l_1 + \phi + C'O.\end{aligned}$$

But $C'O = \sin C'O = \tan\lambda \,.\, \cot SOC'$,

$$\text{and } \cot SOC' = \tan CNO \,.\, \cos NO = p \,.\, \cos(l_1 - \pi);$$

$$\therefore\ l = l_1 + \phi + p \,.\, \tan\lambda \,.\, \cos(l_1 - \pi), \ \ldots\ldots (1),$$

$$\text{and } \lambda = SC - CO = \lambda_1 - p \,.\, \sin(l_1 - \pi) \ldots\ldots (2).$$

Similarly, if l', λ' and p' be values of l, λ and p, for any other time $1750+t'$,

$$l' = l_1 + \phi' + p' \,.\, \tan\lambda' \,.\, \cos(l_1 - \pi') \ldots\ldots\ldots (3),$$

$$\lambda' = \lambda_1 - p' \,.\, \sin(l_1 - \pi') \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

But $\tan\lambda' = \tan\lambda$, nearly, and $l_1 - \pi$ and $l_1 - \pi'$ each nearly equal $l_1 - \frac{1}{2} \,.\, (\pi + \pi')$; therefore calling this quantity L, and eliminating l_1, between (1) and (3), and λ_1 between (2) and (4), we get

$$l' = l + \phi' - \phi + (p' - p) \,.\, \tan\lambda \,.\, \cos L,$$

$$\text{and } \lambda' = \lambda - (p' - p) \,.\, \sin L;$$

therefore substituting in the equations the values of ϕ, p, &c. we get

$$l' = l + (t'-t).\{50''.176068 + (t'+t)\,0''.0001221438\}$$
$$+ (t'-t).\{0''.48892 - (t'+t)\,0''.0000020719\}.\tan\lambda.\cos L,$$
$$\lambda' = \lambda - (t'-t).\{0''.48892 - (t'+t)\,0''.0000030719\}\ \sin L,$$
$$\text{where } L = l_1 - \tfrac{1}{2}(\pi+\pi') = l - \phi - \tfrac{1}{2}.(\pi+\pi'), \text{ nearly}$$
$$= l - 171^0\ 36''\ 10' - 50''.176t + 5''.18.(t+t').$$

The above methods of finding the values of l' and λ' will not apply, unless p, which does not amount to $1'$ in a century, is very small, and for very considerable intervals the relations between l' and l, or between λ' and λ must be found by the common rules of Spherical Trigonometry.

187. From the latitudes, longitudes and obliquity corrected as above, the right ascensions and declinations may be found; but it is obvious from the inspection of the values of l' and λ', that the latitude may be considered as unchanged for moderate intervals, whilst the longitude has an annual increase of $50''.18$, and it is upon this supposition that the precession in right ascension and declination for intervals not exceeding 5 years, is found in the next article.

188. *To find the precession in right ascension and declination of a given star.*

Let P (fig. 33.) be the pole of the equator, K of the ecliptic, S and S' two apparent consecutive positions of a fixed star, after an interval of t years, then the co-latitudes KS, KS' remain unaltered, and SKS' the apparent motion in longitude $= 50''.18t$.

Draw $S'O$ perpendicular to PS, and let SO, the precession in declination $= \alpha$, SPS', the precession in right ascension $= \beta$, and $S =$ the angle of position PSK.

$$\text{Then } \alpha = SS'.\cos S'SO = 50''.18t\,.\sin KS\,.\sin PSK$$
$$= 50''.18t\,.\sin PK\,.\sin KPS$$
$$= 50''.18t\,.\sin\omega\,.\cos a.$$

$$\text{Again, } \beta . \sin PS = S'O = SS' . \sin S'SO$$

$$= 50''.18t . \cos\lambda . \cos S;$$

$$\therefore \beta = \frac{50''.18t . \cos\lambda . \cos S}{\cos\delta},$$

but since $\sin PS . \sin KS . \cos S = \cos PK - \cos PS . \cos KS$;

$$\therefore \cos\delta . \cos\lambda . \cos S$$

$$= \cos\omega - \sin\delta . \{\sin\omega \cos\delta . (-\sin a) + \cos\omega . \sin\delta\}$$

$$= \cos\omega . \cos^2\delta + \sin\delta . \cos\delta . \sin\omega . \sin a;$$

$$\therefore \frac{\cos\lambda . \cos S}{\cos\delta} = \cos\omega + \tan\delta . \sin\omega . \sin a;$$

$$\therefore \beta = 50''.18t . (\cos\omega + \sin\omega . \tan\delta . \sin a).$$

189. Cor. The precession increases the declination of a star whose right ascension is less than 90°, or between 270° and 360°, and diminishes the declinations in the other quadrants. If the latitude and declination of a star are both north or both south, then the precession increases or diminishes the right ascension, according as the cosine of the angle of position is positive or negative. If the angle of position = 90°, or 270°, the precession does not alter the right ascension.

On Nutation.

190. It has been observed, that besides the mean changes of obliquity and precession, the plane of the equator, and consequently its pole, is subject to an irregular motion called Nutation. Theory shews, that if $d\omega$ and dl are the consequent changes in the obliquity and in the longitude of the pole of the equator,

$$dl = -18''.03 \sin(\text{☽} \; \text{☊}) - 1''.13 \sin 2\odot - 0''.22 \sin 2\text{☽} \quad \ldots (1),$$

$$d\omega = \quad 9''.05 . \cos(\text{☽} \; \text{☊}) = 0''.49 . \cos 2\odot - 0''.09 . \cos 2\text{☊} \; \text{☽} \quad ..(2).$$

where $\odot$ ☽, ☽ ☊, represent the longitudes of the Sun, of the Moon, and of the Moon's ascending node, respectively.

If the equations (1) and (2) of Art. 88. be differentiated with respect to a, δ, l and ω, and for dl and $d\omega$, the above values be substituted, the corresponding variations of a and δ, or the nutation in declination and right ascension may be found.

191. *To construct for the nutation.*

It is obvious, that the two first terms of dl and $d\omega$ are of the most importance, and their effect may be conveniently represented by the following construction.

Let $9''.65 = h$, $18''.03 \sin \omega = g$, and round the mean place of the pole of the equator P, (fig. 34.) unaffected by nutation, with semi-axes PA (h), and $PB\,(g)$ describe an ellipse, the major axis of which touches the arc of the solstitial colure. Take $APQ = (360^0 - \text{☽☊})$, and let PQ meet the circle described on the major axis in Q; draw Qo perpendicular to AP, cutting the ellipse in p, then p is the true place of the pole affected by nutation. For k being the pole of the ecliptic,

P's motion in latitude $(d\omega)$

$$= Po \;=\; PQ\,.\cos QPo = h\,.\cos \text{☽☊} = 9''.65\,.\cos \text{☽☊},$$

and P's motion in longitude (dl)

$$= pKo = \frac{po}{\sin Ko} = \frac{g}{h}\,.\,\frac{Qo}{\sin PK}, \text{ nearly,}$$

$$= -\frac{g}{h}\,.\,h\,.\,\frac{\sin \text{☽☊}}{\sin \omega} = -g\,\frac{\sin \text{☽☊}}{\sin \omega} = -\;18''.03 \sin \text{☽☊}.$$

192. *To find the lunar nutation of a given star in right ascension and declination.*

The same construction remaining, let ♈ L (fig 34.) be the ecliptic, and ♈ s, ♈ s' the positions of the equator corresponding to the places of the pole P and p, then ♈ ♈$' = Ll = LKl = P$'s motion in longitude. Draw the declination circles PSs, pSs' through a star S, Ps meeting the equator ♈$'s'$ in t, and draw ♈ v perpendicular to the equator, and mon perpendicular to PS or pS.

Then $Po = PQ\,.\cos QPo = h\,.\cos \text{☽☊}$;

$$po = \frac{g}{h} \cdot Qo = g \sin QPo = -g \cdot \sin \text{☽☊};$$

therefore the nutation in north polar distance

$$\begin{aligned}
&= SP - Sp \\
&= Pm - pn \\
&= Po \cdot \cos oPm - po \cdot \sin oPm \\
&= Po \cdot \sin a + po \cdot \cos a,
\end{aligned}$$

a being the right ascension of S

$$= h \cdot \cos \text{☽☊} \cdot \sin a - g \sin \text{☽☊} \cdot \cos a \ldots (1).$$

The nutation in right ascension

$$= \text{♈}'s' - \text{♈}s = \text{♈}'v + ts'.$$

$$\text{But } \text{♈}'v = \text{♈}\text{♈}' \cdot \cos \omega = Ll \cdot \cos \omega$$

$$= \frac{po}{\sin \omega} \cdot \cos \omega = -g \sin \text{☽☊} \cdot \cot \omega,$$

$$\begin{aligned}
\text{and } ts' &= mn \cdot \frac{\sin \text{dec}}{\cos \text{dec}} \\
&= \tan \delta \cdot (mo + no) \\
&= \tan \delta \, (Po \cdot \sin oPm + po \cdot \cos oPm) \\
&= \tan \delta \, (-Po \cdot \cos a + po \cdot \sin a) \\
&= \tan \delta \cdot (-h \cdot \cos \text{☽☊} \cdot \cos a - g \sin \text{☽☊} \cdot \sin a);
\end{aligned}$$

therefore the whole nutation in right ascension

$$= -g \sin \text{☽☊} \cdot \cot \omega - \tan \delta . (h \cos \text{☽☊} \cdot \cos a + g \sin \text{☽☊} \cdot \sin a) . (2).$$

193. Cor. 1. The first term of the nutation in right ascension, is independent of the star's place, and is the same for all stars, being the quantity by which the equinox recedes in consequence of nutation.

194. Cor. 2. The nutation in north polar distance

$$= -g\cos a\,.\,\left(\sin \text{☽☊} - \frac{h}{g}\,.\tan a\,.\cos \text{☽☊}\right).$$

$$\text{Let } -\frac{h}{g}\tan a = \tan B;$$

therefore the nutation in north polar distance

$$= -\frac{g\cos a}{\cos B}\,.\sin(B + \text{☽☊}) \,\ldots\ldots\ldots\ldots\, (3).$$

The nutation in right ascension

$$= -(g\cot\omega + g\sin a\tan\delta)\sin\text{☽☊} - h\tan\delta\cos a\,.\cos\text{☽☊}.$$

$$\text{Let } h\tan\delta\cos a = g(\cot\omega + \sin a\tan\delta)\,.\tan B';$$

therefore the nutation in right ascension

$$= -g(\cot\omega + \sin a\tan\delta)\,.\,(\sin\text{☽☊} + \tan B'.\cos\text{☽☊})$$

$$= -g(\cot\omega + \sin a\tan\delta)\,.\sin(B' + \text{☽☊}) \,\ldots\, (4).$$

These formulæ are very convenient when the nutation of the same star is required at different times, as the coefficients and the quantities B and B' are always the same.

195. Cor. 3. When the nutation in north polar distance is a maximum,

$$\text{☽☊} = 90^\circ - B.$$

When the nutation in right ascension is a maximum,

$$\text{☽☊} = 90^\circ - B'.$$

196. Cor. 4. The change of the equinoxial point in longitude ♈♈′

$$= -18''.03\,.\sin\text{☽☊}. \quad \text{(Art. 191.)}$$

And the change in right ascension ♈♈′ $\cos\omega$

$$= -18''.03\,.\cos\omega\,.\sin\text{☽☊}.$$

the apparent deviation of the star from its true place, is called the *Aberration*.

201. *To find the aberration of a fixed star.*

Complete the parallelogram SA (fig. 35.); then

sin aberration : sin AsS :: Ss : SA

:: vel. of the Earth : vel. of light.

But $AsS = sAC = SAC$ nearly, the angle which a line joining the Earth and the star makes with the direction of the Earth's motion, which angle is called the *Earth's way;* and sin aberration = aberration nearly;

$\therefore$ the aberration : sin $\oplus$'s way :: vel. $\oplus$: vel. light.

Now light is $8'\,13''$ in passing over a radius of the Earth's orbit (r), and in this time the Earth describes an angle of $20''.25$ round the Sun, or an arc $= 20''.25\,r$;

$\therefore$ the aberration : sin $\oplus$'s way :: $20''.25.r$: r;

$\therefore$ the aberration $= 20''.25$. sin $\oplus$'s way.

202. The aberration of a fixed star at any time takes place in a plane passing through the star and a tangent to the curve described by the Earth, and always tends towards those parts in the direction of which the Earth moves; therefore, if the Earth move towards the star, or the angle SAC be acute, the star appears depressed towards the plane of the ecliptic. If the Earth move from the star, it appears more elevated in consequence of aberration.

203. It is most convenient, in treating of the effects of aberration on the apparent places of the stars, to refer them to the surface of a sphere having the Sun in its center, and a radius equal to the distance of the fixed stars. As the plane in which the aberration takes place does not pass through the Sun, its intersection with the surface of the sphere is not a great circle,

but if we suppose a plane to pass through the Sun and the true and apparent places of a fixed star, this plane may be considered parallel to the former, on account of the immense distance of the stars, and the aberration and Earth's way may be considered without sensible error, as measured on the arcs of the great circle in which this plane cuts the surface of the sphere. The plane thus drawn being parallel to the direction of the Earth's motion, cuts the ecliptic in two points 90^0 distant from the Earth, if the Earth's orbit be supposed circular.

204. If S (fig. 35.) be the true, and s the apparent place of a fixed star, Ss is parallel to the direction in which the Earth moves, and, therefore, to the ecliptic, and is to AS, the distance of the star from the Earth :: vel. Earth : vel. light. The velocity of light is constant, and that of the Earth may be supposed uniform, since the Earth's orbit, as will be shewn in the next Chapter, is nearly circular; also, AS may be considered constant and parallel to itself in all positions of the Earth, because the diameter of the Earth's orbit subtends no finite angle at S; therefore, Ss is constant, and s describes a circle round S, in a plane parallel to the ecliptic.

This circle is seen from the Earth, projected on a plane perpendicular to AS, and will therefore appear elliptical, unless AS be perpendicular to its plane, or S be situated in the pole of the ecliptic. The angle $SAs = 20''.25$. sin Earth's way, the greatest value of which is $20''.25$, and the least takes place when the Earth's way = the acute angle which SA makes with the plane of the ecliptic, or the star's latitude; therefore, the major axis of the ellipse, which a star appears to describe about its real place, in consequence of aberration,

$$= 40''.50;$$

and the minor axis $= 40''.50$. sin star's latitude.

DEF. The curve which a star appears to describe about its true place is called the aberratic curve.

205. *To find the equation to the aberratic curve, upon any hypothesis of the form of the Earth's orbit.*

Let r be the radius vector, and p the perpendicular let fall from the Sun on the tangent at any point of the Earth's orbit, ρ and π the distance and perpendicular for the corresponding point of the aberratic curve, the true place of the star being taken for the pole. Then ρ varies as the Earth's velocity, or as $\frac{1}{p}$; let it equal $\frac{c^2}{p}$. Also, since ρ is always parallel to the tangent of the Earth's orbit, its angular velocity is the same as that of the tangent, and, therefore, of p. But the angular velocity of ρ

$$= \frac{\pi d\rho}{\rho\sqrt{\rho^2 - \pi^2}} = -\frac{\pi dp}{\sqrt{c^4 - \pi^2 p^2}},$$

substituting for ρ its value $\frac{c^2}{p}$; and the angular velocity of p

$$= \frac{dp}{\sqrt{r^2 - p^2}};$$

$$\therefore \frac{dp}{\sqrt{r^2 - p^2}} = -\frac{\pi dp}{\sqrt{c^4 - \pi^2 p^2}};$$

$$\therefore c^4 - \pi^2 p^2 = \pi^2 r^2 - \pi^2 p^2;$$

$$\therefore c^4 = \pi^2 r^2, \text{ and } r = \frac{c^2}{\pi}.$$

Therefore, substituting in the equation between r and p their values $\frac{c^2}{\rho}$ and $\frac{c^2}{\pi}$, the equation between ρ and π is obtained. The equation thus found may, if required, be transformed by the common methods into an equation between rectangular or polar co-ordinates.

206. Ex. Suppose the Earth's orbit an ellipse, of which the axes are a and b; then, the equation between r and p is

$$p^2 = \frac{b^2 r}{2a - r}, \quad \text{or } r = \frac{2ap^2}{b^2 + p^2};$$

$$\therefore \ \pi = \frac{c^2}{r} = \frac{c^2}{2a} \cdot \frac{b^2 + p^2}{p^2}$$

$$= \frac{c^2}{2a} \cdot \frac{b^2 \rho^2 + c^4}{c^4}, \text{ since } p = \frac{c^2}{\rho},$$

$$= \frac{b^2 \rho^2 + c^4}{2ac^2}.$$

We may easily shew that this is the equation of a circle; for let S (fig. 36.) be a point in a circle rad $CP\,(m)$, at a distance $SC\,(n)$ from the center, and taking any point P in the circumference, let $SP = \rho$, $SY = \pi$,

$$\therefore \ n^2 = \rho^2 + m^2 - 2m\rho \cdot \cos SPC$$

$$= \rho^2 + m^2 - 2m\pi;$$

$$\therefore \ \pi = \frac{m^2 - n^2 + \rho^2}{2m},$$

which coincides with the aberratic curve, if

$$m = a \cdot \frac{c^2}{b^2}, \quad \text{and } m^2 - n^2 = \frac{c^4}{b^2}.$$

Hence it appears that the stars describe circles in consequence of aberration, even if the Earth's elliptic motion be taken into consideration. As in Art. 204, unless the star be situated in the pole of the ecliptic, the plane of the circle, being viewed obliquely, will appear an ellipse.

207. *To find the aberration of a star with respect to any plane.*

Let E (fig. 37. and 38.) be the Earth in the ecliptic EC, S the star, IS the plane, to or from which the aberration is required, which is called the *relative* plane, OS the great circle in which the aberration takes place; therefore, SO = the Earth's way, and $EO = 90^0$. Draw $s\sigma$ perpendicular to OS, then, $s\sigma$ is the deflection from the plane IS caused by aberration, and

$$\begin{aligned} s\sigma &= Ss \,.\, \sin ISO \\ &= 20''.25 \sin SO \,.\, \sin ISO \\ &= 20''.25 \,.\, \sin OI \,.\, \sin I \\ &= \pm 20''.25 \,.\, \sin I \,.\, \cos IE. \end{aligned}$$

+ or − according as EI is less or greater than 90^0.

208. *To find the aberration of a given star in longitude.*

Let SI (fig. 39.) be a circle of latitude passing through a star S, P the pole of the ecliptic; therefore, PI and the angle I each $= 90^0$. Let, as before, λ = the latitude, l = the longitude, δ = the declination, a = the right ascension of the star S, and let SS' be the aberration perpendicular to PI; then, if $\odot$ be the longitude of the Sun seen from the Earth, $180^0 + \odot$ will be the longitude of the Earth seen from the Sun ($\oplus$), because the Earth and Sun, seen from each other, are in opposite parts of the ecliptic. Therefore, we have

$$SS' = 20''.25 \,.\, \sin I \,.\, \cos IE,$$

but $\sin I = 1$,

and $\cos IE = \cos(\oplus - l) = \cos(180^0 + \odot - l) = -\cos(\odot - l)$;

$$\therefore\ SS' = -20''.25 \,.\, \cos(\odot - l),$$

and the aberration in longitude

$$= -\frac{20''.25 \,.\, \cos(\odot - l)}{\cos\lambda}.$$

209. *To find the aberration in latitude.*

Let SI (fig. 40.) be perpendicular to the circle of latitude PSL, passing through a given star S; then, I is the pole of

SL; therefore, $IL = 90^0$, and the angle $SIL = \lambda$; therefore, if SS' = the aberration in latitude,

$$SS' = -20''.25 \,.\, \sin I \,.\, \cos IE,$$

$$\text{but } \cos IE = -\sin EL = -\sin(180 + \odot - l) = \sin(\odot - l),$$

$$\text{and } \sin I = \sin \lambda;$$

therefore, the aberration in latitude

$$= -20''.25 \,.\, \sin \lambda \,.\, \sin(\odot - l).$$

210. *To find the aberration in right ascension.*

Let SI (fig. 41.) be a circle of declination meeting the equator in s and the ecliptic in I, and let SS' be the aberration perpendicular to IS; then, drawing the declination circle $PS's'$ through S', and making $\Upsilon I = \phi$, where ϕ is determined by the equation $\cot \phi = \cos \omega \,.\, \cot a$, we have

$$SS' = 20''.25 \,.\, \sin I \,.\, \cos IE,$$

$$\text{but } \sin a = \sin \phi \,.\, \sin I,$$

$$\text{and } \cos IE = \cos(\oplus - \phi) = \cos(180 + \odot - \phi) = -\cos(\odot - \phi);$$

$$\therefore\ SS' = -20''.25 \,.\, \frac{\sin a \,.\, \cos(\odot - \phi)}{\sin \phi};$$

therefore, the aberration in right ascension ss'

$$= -\frac{20''.25 \,.\, \sin a \,.\, \cos(\odot - \phi)}{\sin \phi \,.\, \cos \delta}.$$

Cor. If $\odot - \phi = 90^0$ or 270^0, that is, if the Sun's longitude $= 90^0 + \phi$, or $270^0 + \phi$, there is no aberration in right ascension.

211. *To find the aberration in declination.*

Let the relative plane IS (fig. 42.) be perpendicular to the circle of declination *SD*, and let the equator ♈ *Q* meet these circles in *M* and *D*, then, *M* is the pole of *SD*, and $SMD = SD = \delta$; therefore, in the triangle *MI*♈, we have

$$M♈ = 90^0 - a, \quad M♈ I = \omega, \quad IM♈ = 180^0 - \delta,$$

from which data *I*♈ (θ) may be found by the usual methods. Also, in the same triangle,

$$\sin\theta . \sin I = \sin M♈ . \sin M = \cos a . \sin\delta;$$

therefore, the aberration in declination *SS*′

$$= -20''.25 . \sin I . \cos EI$$

$$= -20''.25 . \frac{\cos a . \sin\delta}{\sin\theta} . \cos(180 + \odot + \theta),$$

$$= 20''.25 . \frac{\cos a . \sin\delta . \cos(\odot + \theta)}{\sin\theta}.$$

Cor. If the Sun's longitude $= 90^0 - \theta$ or $270^0 - \theta$, the aberration in declination vanishes.

In the four cases for which the aberration has been deduced, the Earth has been placed in a position to increase the latitudes, longitudes, &c.; therefore, the aberrations, with their proper signs, must be all added to the true latitudes and longitudes, to obtain the apparent values of the same quantities. In the second and fourth cases, *EI* has been taken greater than 90°. These observations require attention in order to obtain the right sign of the aberration.

The subsidiary angles ϕ and θ have been introduced in the two last articles to facilitate the logarithmic calculation of the aberrations; there is, however, no difficulty in obtaining formulæ depending on the right ascensions and declinations only.

212. *To find the aberration in right ascension and declination, in terms of a and δ.*

The aberration in right ascension $= \dfrac{SS'}{\cos\delta}$ (fig. 41.)

$$= 20''.25 \,.\, \frac{\sin I \,.\, \cos IE}{\cos\delta}\,.$$

$$= \frac{20''.25}{\cos\delta} \,.\, \sin I \,.\, \cos(\Upsilon E - \Upsilon I)$$

$$= \frac{20''.25}{\cos\delta} \,.\, \{\cos\Upsilon E \,.\, \sin I \,.\, \cos\Upsilon I + \sin\Upsilon E \,.\, \sin I \,.\, \sin\Upsilon I\}.$$

but $\sin I \,.\, \sin\Upsilon I = \sin a$;

$$\therefore\ \sin I \,.\, \cos\Upsilon I = \sin I \,.\, \sin\Upsilon I \,.\, \cot\Upsilon I$$

$$= \sin a \,.\, \cos\omega \,.\, \cot a$$

$$= \cos\omega \,.\, \cos a,$$

and $\Upsilon E = 180^0 + \odot$;

therefore, the aberration in right ascension

$$= \frac{20''.25}{\cos\delta} \,.\, \{-\cos\odot \,.\, \cos\omega \,.\, \cos a - \sin\odot \,.\, \sin a\}$$

$$= -\frac{20''.25}{\cos\delta} \{\cos a \,.\, \cos\omega \,.\, \cos\odot + \sin a \,.\, \sin\odot\}.$$

The aberration in declination SS' (fig. 42.)

$$= -20''.25 \,.\, \sin I \,.\, \cos EI$$

$$= -20''.25 \,.\, \sin I \,.\, \cos(I\Upsilon + \Upsilon E)$$

$$= -20''.25 \,.\, \{\cos\Upsilon E \,.\, \sin I \,.\, \cos I\Upsilon - \sin\Upsilon E \,.\, \sin I \,.\, \sin I\Upsilon\};$$

but $\sin I \,.\, \sin I\Upsilon = \sin\delta \,.\, \cos a$;

$$\therefore\ \sin I \,.\, \cos I\Upsilon = \sin I \,.\, \sin I\Upsilon \,.\, \cot I\Upsilon$$

$$= \sin\delta \,.\, \cos a \,.\, \cot\theta.$$

Also in the triangle $I\text{♈}M$,

$$\sin a \,.\, \cos \omega = \cot \theta \,.\, \cos a - \sin \omega \,.\, \cot (180^\circ - \delta);$$

$$\therefore \cot \theta = \tan a \,.\, \cos \omega - \frac{\sin \omega \,.\, \cot \delta}{\cos a};$$

$$\therefore \sin I \,.\, \cos I\text{♈} = \sin \delta \,.\, \sin a \cos \omega - \cos \delta \sin \omega;$$

therefore the aberration in declination

$$\begin{aligned} &= -20''.25 \,\{ -\cos \odot \,.\, (\sin \delta \,.\, \sin a \cos \omega \\ &\qquad - \cos \delta \sin \omega) + \sin \odot \,.\, \sin \delta \,.\, \cos a \} \\ &= \;\; 20''.25 \,\{ \sin \delta \,.\, (\sin a \cos \omega \cos \odot \\ &\qquad - \cos a \sin \odot) - \cos \delta \,.\, \sin \omega \,.\, \cos \odot \}. \end{aligned}$$

213. *To find the aberration taking into consideration the elliptic form of the Earth's orbit*[*].

As in Art. 207. let E (fig. 43.) be the Earth's place referred to the ecliptic, S a fixed star, SI the relative plane, ρ the distance of the Earth from the Sun, v the true, and m the mean anomaly; then we may suppose the Earth's velocity compounded of the velocity ρdv, in consequence of which it describes an angle round the Sun, or an arc of the ecliptic, and of the velocity $d\rho$, by which it moves from the Sun. The effect of the former relative to the plane SI, has been found $= \dfrac{\text{vel} \oplus}{\text{vel light}} \,.\, \sin I . \cos IE$, $= \rho dv \,.\, \sin I \,.\, \cos IE$, supposing the mean distance of the Sun and Earth unity. But since aberration always takes place in a plane passing through the star parallel to the direction in which the Earth moves, the effect of the Earth's velocity $d\rho$ takes place in the plane SE, and if Ss represent it,

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[*] This article depends on the formulæ of the succeeding chapter, and requires the previous perusal of it.

$$Ss = d\rho \,.\, \sin SE\,;$$

therefore the aberration from SI (σs)

$$= d\rho \,.\, \sin SE \,.\, \sin ESI$$

$$= d\rho \,.\, \sin I \,.\, \sin IE,$$

and the whole aberration

$$= \rho dv \,.\, \sin I .\, \cos IE + d\rho \,.\, \sin I \,.\, \sin IE \ldots\ldots (1).$$

Now $\rho^2 dv = (1-e^2)^{\frac{1}{2}} dm\,;$

$$\therefore\ \rho dv = (1-e^2)^{\frac{1}{2}} .\, dm \,.\, \frac{1+e\cos v}{1-e^2}$$

$$= dm \,.\, (1+e\cos v),\ \text{neglecting the square of } e$$

$$= 20''.25 \,.\, (1+e\cos v),\ \text{nearly.}$$

But if Π be the longitude of the perigee,

$$v = \oplus - \text{longitude of the perihelion}$$

$$= 180 + \odot - (180 + \Pi) = \odot - \Pi\,;$$

$$\therefore\ \rho dv = 20''.25 \,.\, \{1 + e\cos(\odot - \Pi)\},$$

$$\text{and } d\rho = d \,.\, \frac{1-e^2}{1+e\cos v} = d .\, (1 - e\cos \nu)$$

$$= e \sin v . dv$$

$$= 20''.25 . e \sin(\odot - \Pi)\ \text{very nearly;}$$

therefore substituting the values of ρdv and ρ in (1), the whole aberration

$$= 20''.25 \,.\, \{\sin I \,.\, \cos IE \,.\, [1 + e\cos(\odot - \Pi)]$$

$$+ \sin e \,.\, \sin(\odot - \Pi) \,.\, \sin I \,.\, \sin IE\}.$$

214. The aberration caused by the eccentricity

$$20''.25 \,.\, e \,.\, \sin I \,.\, \{\cos IE \,.\, \cos(\odot - \Pi) + \sin IE \,.\, \sin(\odot - \Pi)\}$$

$$= 20''.25 e \,.\, \sin I .\, \cos(IE + \Pi - \odot),$$

which cannot exceed $20''.25 e$ or $0''.25$, and may therefore be neglected. If, however, it should be thought useful to have the aberrations caused by the eccentricity, they may be easily obtained from the preceding values of I and IE. The aberrations thus deduced by Delambre, (*Astronomie Lecon.* XIX.) are

aberration in longitude

$$= -\,0''.34\,.\cos(\Pi - l)\,.\,\mathrm{sect}\,\lambda,$$

aberration in latitude

$$= 0''.34\,.\sin\lambda\,.\sin(\Pi - \lambda),$$

aberration in right ascension,

$$= -\,0''.34\,.(\cos\omega\,.\cos a\,.\cos\Pi + \sin a\,.\sin\Pi)\,.\sec\delta,$$

aberration in declination

$$= 0''.34\,.\left\{\sin\delta\,.(\cos\omega\,.\sin a\,.\cos\Pi - \cos a\,.\sin\Pi)\right.$$

$$\left. - \cos\delta\,.\sin\omega\,.\cos\Pi\right\}.$$

215. *To find the coefficient of aberration from observations of the stars without previous knowledge of the velocity of light.*

Let a star in or near the solstitial colure be observed at the time of the vernal and autumnal equinox, and let the difference of declination at these times caused by aberration $= D$; then for this star $\sin a$ may be taken $= 1$, and $\cos a = 0$; therefore, denoting the required coefficient of aberration by x, we have, since $\odot = 0$ at the first, and 180 at the second observation, the aberration at

$$\text{the vernal equinox} = \quad x\,.(\sin\delta\,.\cos\omega - \sin\omega\cos\delta),$$

$$\text{autumnal} \ldots\ldots\ldots = -\,x(\sin\delta\cos\omega - \sin\omega\cos\delta);$$

$$\therefore\ D = 2x\sin(\delta - \omega);$$

$$\therefore\ x = \tfrac{1}{2}\,.\,\frac{D}{\sin(\delta - \omega)}.$$

216. The tables of aberration may be made use of in calculating the values of the annual parallax of the fixed stars, as

the formulæ by which both these quantities are expressed, have a close analogy. If $180^0 + \odot$ be substituted for $\oplus$ in Art. 179, and the parallaxes are considered additive to the true latitudes and longitudes, we have

$$\text{the parallax in longitude} = -\phi \cdot \frac{\sin(l - \odot)}{\cos\lambda},$$

$$\therefore \text{aberration} \ldots\ldots\ldots = -20''.25 \cdot \frac{\cos(l - \odot)}{\cos\lambda},$$

$$\text{the parallax in latitude} = -\phi \cdot \sin\lambda \cdot \cos(l - \odot),$$

$$\therefore \text{aberration} \ldots\ldots\ldots = -20''.25 \cdot \sin\lambda \cdot \sin(l - \odot).$$

Let α and β be the aberrations taken out of the tables of aberration for the Sun's longitude $90 + \odot$;

$$\therefore \alpha = -20''.25 \cdot \frac{\sin(l - \odot)}{\cos\lambda},$$

$$\beta = -20''.25 \cdot \sin\lambda \cos(l - \odot);$$

$$\therefore \text{the parallax in longitude} = \frac{\phi}{10''.25} \cdot \alpha,$$

$$\ldots\ldots\ldots\ldots \text{latitude} = \frac{\phi}{10''.25} \cdot \beta.$$

The annual parallax in right ascension and declination may be deduced in a similar manner from the aberration in right ascension and declination.

From the comparison of the expressions for the aberration and parallax it appears, that the parallax is greatest when the aberration is least, and conversely.

217. The diurnal motion of the Earth combined with the motion of light will also cause a small change in the places of the stars, but too small to have any observable influence, except perhaps on the right ascension of stars near the pole. The coefficient of this aberration for a place whose latitude is l

$$=\frac{\text{vel of the place}}{\text{vel of light}}=\frac{\text{vel of equator . cos } l}{\text{vel of light}}=0''.3 \cos l\text{, nearly.}$$

218. *To find the aberration in right ascension and declination caused by the diurnal rotation.*

Let IE (fig. 39.) be the equator, P its pole, E' the place of an observer's zenith, referred to the equator in E, PSI the declination circle through a star S, SS' the deflection from the declination circle caused by aberration. Then, as in Art. 208. the aberration in right ascension

$$=\frac{SS'}{\cos\delta}$$

$$=\frac{0''.3 \,.\, \cos l}{\cos\delta}\,.\,\sin I \,.\, \cos IE$$

$$=\frac{0''.3 \cos l \,.\, \cos h}{\cos\delta},$$

h being the hour angle IPE.

Next, let IS (fig. 40.) be a circle perpendicular to the declination circle PS, and E, as before, the place in which the observer's meridian meets the equator; then the aberration in declination

$$= SS'$$

$$= -\ 0''.3 \cos l \,.\, \sin I \,.\, \cos IE$$

$$= 0''.3 \cos l \sin\delta \,.\, \sin h,$$

since $EI = 90^0 + h$.

219. *To find the effect of aberration on the Sun, Moon and planets.*

Let p and e (fig. 44.) be cotemporary positions of the Earth and a planet, P and E other cotemporary positions after an interval (t'') in which light moves from p to e. Then if the Earth were at rest at E, the planet would be seen by the ray

pE, emitted when the planet was at p. Take now $Ee' = Ee$, and complete the parallelogram Eq, then pEq is the aberration caused by the Earth's motion; therefore the whole aberration $=$ PEq, or the planet, when at P, will be seen in the direction Eq. But $PEq = PEp + pEq = PEp + Epe =$ the motion of the planet round E at rest $+$ the motion of E round p at rest $=$ the whole geocentric motion of the planet in t'', which, if m be the geocentric motion of the planet in $1'' = mt$. Now if D be the planet's distance from the Earth, considering the distance of the Earth from the Sun unity,

$$1 : D :: 8'\ 13'' : t = D . 8'\ 13''.$$

$$\text{Therefore the aberration} = m . 8'\ 13''\ D.$$

This aberration may be resolved in any direction, and will preserve the same ratio with the planet's geocentric motion resolved in the same directions, so that

$$T = A - m\, 8'\ 13''. D,$$

where T is the true and A the apparent latitude, longitude, right ascension or declination of P, and m the increment of the same quantities in $1''$.

Fig. 44. represents the planet's motion retrograde, if the motion be direct as in fig. 45. $PEq = PEp - Epe =$, as before, the geocentric motion of P round E in t''.

220. The apparent places of the fixed stars are found to coincide so exactly with the places deduced from the formulæ of aberration as to leave no doubt of the truth of the hypotheses upon which the formulæ are constructed. This affords the most convincing proof both of the motion of light and of the real motion of the Earth round the Sun.

CHAP. XII.

THE THEORY OF ELLIPTIC MOTION.

221. The path of the Earth round the Sun, or (which has been shewn to be the same thing) the apparent path of the Sun round the Earth, lies in the plane of the ecliptic. This plane has its position in space sufficiently determined by the latitudes of known fixed stars, and has been determined with reference to the equator, by the obliquity and place of the equinox.

222. If the Sun's path in the ecliptic were a circle having the Earth in its center, the Sun's apparent diameter would be the same at all times of the year, but the Sun's diameter varies in the course of a year, from the greatest $16' 18''$ to the least $15' 45''$, the maximum and minimum taking place at an interval of half a year, and after a motion of 180^0 of longitude.

223. The Sun's diameter, on account of the greater accuracy of the method, is not observed directly, but is calculated from the observed time of transit of the Sun's disk over the meridian. If t be this time, and T the time between two successive transits, $T : t :: 360^0 :$ Sun's diameter, supposing the Sun to be in the equator. When the Sun is not in the equator, since the arc it describes is part of a small circle, the diameter found as above must be diminished in the proportion of cosine of declination : radius.

224. *The curve which the Earth describes round the Sun is an ellipse.*

Let Δ and Δ' be the greatest and least apparent diameters of the Sun, δ any other diameter, $a+ae$, $a-ae$, and ρ the corresponding distances of the Earth and Sun, and l, $l+180$, $l+v$ the corresponding longitudes of the Sun. Then, by comparing the observed longitudes and diameters of the Sun at different times, it appears that $\Delta-\delta$, the decrement of the Sun's diameter, varies as the versed-sine of the difference of the Sun's longitudes, or as $1-\cos v$;

$$\therefore\ \Delta-\Delta' : \Delta-\delta :: 1-\cos 180 : 1-\cos v;$$

$$\therefore\ (\Delta-\Delta')\,.\,(1-\cos v)=2\Delta-2\delta;$$

$$\therefore\ \delta=\Delta-\tfrac{1}{2}\,.\,(\Delta-\Delta')\,.\,(1-\cos v)$$

$$=\tfrac{1}{2}(\Delta+\Delta')+\tfrac{1}{2}\,.\,(\Delta-\Delta')\,.\,\cos v;$$

or, substituting for δ, Δ, Δ', the reciprocals of ρ, $a-ae$, $a+ae$, to which they are proportional,

$$\frac{1}{\rho}=\frac{1}{a\,.\,(1-e^2)}+\frac{e}{a\,.\,\overline{1-e^2}}\,.\,\cos v.$$

This is the well known equation of an ellipse having the Earth in its focus, and whose major axis passes through the places of the greatest and least apparent diameters. The extremities of this axis are called, with reference to the Earth's motion round the Sun, *Aphelion* and *Perihelion*, and with reference to the Sun's apparent motion round the Earth, *Apogee* and *Perigee*.

DEF. The angle described by the Earth round the Sun from perihelion in any time t, is called the *True Anomaly* (v). If a body revolve round the Sun with a uniform angular velocity in the time of the Earth's motion round the Sun, the angle described in the same time t is called the *mean anomaly* (m); therefore, if P be the time of the Earth's period, or one year,

$$m=360^0\,.\,\frac{t}{P}.$$

225. The anomalies are often reckoned from the aphelion instead of the perihelion, and any expressions in which they occur, may be transformed from the one to the other, by substituting for v and m $180^0 - v$ and $180^0 - m$.

226. Having calculated the Sun's longitude on several successive days from the observed declinations, the increment of longitude, or the increment of true anomaly in a given small time, is found to vary as the square of the Sun's apparent diameter, or inversely as the square of the distance. If, then, dv be the increment of true anomaly in the time dt, dv varies as $\frac{1}{\rho^2}$; therefore, $\rho^2 dv$ is constant; but $\frac{1}{2}\rho^2 dv$ is the area described by the Earth round the Sun in the given time dt; therefore, as this is constant, the area described by the Earth round the Sun in any time varies as the time.

If the planets' places are calculated upon the hypothesis of their moving in the same manner as the Earth round the Sun, describing equal areas in equal times, the observed places agree with the calculations, which proves the truth of the hypothesis.

The squares of the periodic times of the Earth and planets round the Sun, are found by observation to vary as the cubes of the mean distances. This, together with the elliptic motion of the planets, and the equal description of areas in equal times, form the three laws, originally discovered by Kepler, and on which Newton's theory of universal gravitation mainly depends.

227. *To find the motion of the aphelion.*

The Earth's motion in longitude is the same at equal distances from the aphelion; if therefore L be any longitude of the Earth, before, and L' the longitude after passing the aphelion when the Earth's motion (or the Sun's apparent motion) is the same as before, $\frac{1}{2}.(L' + L)$ is the longitude of the aphelion, supposing the equinox from which L and L' are measured to remain fixed during the interval. If p be the precession of the equinoxes whilst the Earth's longitude increases from L to L',

$\frac{1}{2}.(L' - p + L)$ is the longitude of the aphelion. From comparing the longitude of the aphelion determined in this manner, with its value found after an interval of four or five centuries, it has been found that the longitude of the aphelion has an annual increase of 62″, from which, subtracting the precession, there remains 12″ nearly for the annual motion of the aphelion.

Although the above method does not ascertain the place of the aphelion very correctly, on account of the difficulty of observing when the Sun's motions are exactly equal, yet, on account of the great interval between the two determinations of the aphelion, the motion of that point is determined with great accuracy.

228. *To find the longitude of the perihelion, and the time of the Earth's passing through it.*

Let EAP (fig. 46.) be the Earth's orbit round the Sun S, AP the line of apsides, E the Earth *near* the aphelion at the time T, E' the Earth near the perihelion at the time T', Aa, Pp the motion of A and P whilst the Earth moves from E to E'. Now, the Earth's daily motions in longitude at E and E' being near, the maximum and minimum may be considered uniform; if, therefore, they be represented by a and p, and x and y be the respective times of the Earth's moving from E to A, and from E' to p, $ASE = ax$, $E'Sp = py$. Let $ES♈$ the longitude of $E = L$, $E'S♈'$ the longitude of $E' = L'$; therefore, since the time from E to E' is half a year nearly, $♈S♈' = 25''$, and we have

the longitude of the Earth at $A = L + ax$,

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots P = \text{long. at } p - 6''$$

$$\doteq L' + py - 6'';$$

$$\therefore \text{ long. at } P - \text{long at } A = L' - L + py - ax - 6'',$$

or $180^0 + 25'' = L' - L + (p - a) \cdot y + a \cdot (y - x) - 6''$;

$$\therefore y = \frac{180^0 + 31'' + L - L' - a \cdot (y - x)}{p - a}.$$

But $182^d\ 15^h\ 6'\ 59''$, which is half the time of an anomalistic year (M), $=$ the time of the Earth's moving from perihelion to aphelion

$$= T' + y - (T + x) = T' - T + (y - x);$$

$$\therefore y - x = M - T' + T,$$

and $$y = \frac{180^0 + 31'' + L - L' - a \cdot (M - T' + T)}{p - a};$$

therefore, the longitude of the perihelion at the time T'

$$= L' + py,$$

and the time of the Earth's being in the perihelion

$$= T' + y,$$

are both determined.

COR. 1. The longitude of the perigee $= L' + py - 180$.

COR. 2. If π be the longitude of the perihelion, and l the longitude of the Earth seen from the Sun, or, as it is called, the *heliocentric* longitude of the Earth, then, the corresponding true anomaly $v = l - \pi$.

229. *To find the eccentricity of the orbit described by the Earth round the Sun.*

Let e be the eccentricity, considering the semi-major axis unity; therefore, (a and p being as in the last Article) $\frac{1}{2} \cdot (1 + e)^2 a$ and $\frac{1}{2} (1 - e)^2 \cdot p$ are the areas described round the Sun in equal times, when the Earth is in the perihelion and aphelion, and, as these are equal,

$$\therefore (1+e)\sqrt{a} = \overline{1-e}\,.\sqrt{p};$$

$$\therefore e = \frac{\sqrt{p}-\sqrt{a}}{\sqrt{p}+\sqrt{a}}.$$

230. 2d. method. e may also be determined from the greatest and least apparent diameters of the Sun, if these are Δ and Δ',

$$1+e : 1-e :: \Delta : \Delta';$$

$$\therefore e = \frac{\Delta - \Delta'}{\Delta + \Delta'}.$$

231. *To find the mean distance of the Earth from the Sun.*

Let a be the mean distance of the Earth from the Sun, ρ any other distance, v the corresponding true anomaly, measured from perihelion; then, if l be the heliocentric longitude of the Earth deduced from an observed longitude of the Sun, and π the longitude of the perihelion,

$$\rho = \frac{a\,.\,\overline{1-e^2}}{1+e\,.\cos(l-\pi)}.$$

In this equation e, l, and π are known; therefore, if any distance ρ of the Sun from the Earth could be found, a would be known. A mode of determining accurately the Sun's parallax will be explained in a subsequent Chapter, and from this ρ, and, therefore, a may be obtained.

232. *In elliptical orbits of small eccentricity, the angular motion round the farther focus is nearly uniform.*

Let P (fig. 47.) describe an ellipse of small eccentricity round S, and let H be the other focus; let $PHA = \phi$, and let a, e, v be as before.

Then $\dfrac{a\,.\,\overline{1-e^2}}{1+e\cos v}+\dfrac{a\,.\,\overline{1-e^2}}{1-e\cos\phi}=SP+HP=2a;$

$$\therefore\ \frac{a.\overline{1-e^2}.e\sin v}{(1+e\cos v)^2}\,.\,\frac{dv}{dt}-\frac{a\,.\,\overline{1-e^2}\,.\,e\sin\phi}{(1-e\cos\phi)^2}\,.\,\frac{d\phi}{dt}=0.$$

Or $SP^2\,.\sin v\,.\,\dfrac{dv}{dt}-HP^2\,.\sin\phi\,.\,\dfrac{d\phi}{dt}=0;$

$$\therefore\ SP^2\,.\,\frac{dv}{dt}=HP^2\,.\,\frac{\sin\phi}{\sin v}\,.\,\frac{d\phi}{dt}.$$

But $SP^2\,.\,\dfrac{dv}{dt}$ ($=2$.area described round S in dt) is constant,

and $\dfrac{\sin\phi}{\sin v}=\dfrac{SP}{HP}$.

$\therefore\ SP\,.\,HP\,.\,\dfrac{d\phi}{dt}$ is constant;

$\therefore\ \dfrac{d\phi}{dt}\propto\dfrac{1}{SP\,.\,HP}\propto\dfrac{1}{a^2-x^2}$, if $SP=\overline{a+x}$, and $HP=\overline{a-x}$

$$\propto\frac{1}{a^2}-\frac{x^2}{a^4}+\&c.$$

$$\propto\frac{1}{a^2}\text{ nearly,}$$

since x is very small compared with a. Hence, the motion round H is nearly uniform, and much more so than that round S $\left(\dfrac{dv}{dt}\right)$, which varies as $\dfrac{1}{SP^2}$, or as $\dfrac{1}{\overline{a+x}|^2}$, that is, as $\dfrac{1}{a^2}-\dfrac{2x}{a^3}$ nearly.

233. *Given the mean, to find the true anomaly, in orbits of small eccentricity.*

The same construction remaining, take PHA (fig. 47.) for the mean anomaly (m), produce HP to D, making $PD = PS$, or $HD = 2a$, and join SD.

Then

$$HD + HS : HD - HS :: \tan \tfrac{1}{2}.(HSD + HDS) : \tan \tfrac{1}{2}.(HSD - HDS).$$

$$\text{Or } 2a + 2ae : 2a - 2ae :: \tan \tfrac{1}{2}.DHB : \tan \tfrac{1}{2}.HSP;$$

$$\text{since } PSD = HDS.$$

$$\text{Or } \overline{1+e} : \overline{1-e} :: \tan \tfrac{1}{2}.(180 - m) : \tan \tfrac{1}{2}.(180 - v)$$

$$:: \cot \tfrac{1}{2} m : \cot \tfrac{1}{2} v;$$

$$\therefore \tan \tfrac{1}{2}(v) = \frac{1+e}{1-e}.\tan \tfrac{1}{2} m.$$

This equation determines v very nearly from m in orbits of small eccentricity.

234. If $PSB = v$, and $PHB = m$, the equation becomes

$$\tan \tfrac{1}{2} v = \frac{1-e}{1+e}.\tan \tfrac{1}{2} m,$$

which is the same as would have been deduced by changing e into $-e$. Any other expressions depending on the anomalies reckoned from perihelion, may be changed in the same manner into equivalent expressions, where these quantities are measured from aphelion, as is obvious, since the equation

$$\rho = \frac{a.\overline{1-e^2}}{1+e\cos v}$$

$$\text{becomes } \rho = \frac{a.\overline{1-e^2}}{1-e\cos v}.$$

upon changing e into $-e$, which is the same as if v had been changed into $180 - v$.

235. *To find the relation between the true and mean anomaly.*

Let m be the mean anomaly at any time t (t being expressed in days) from perihelion, then, P being $365^d.2542$, we have

$$m = \frac{2\pi}{P} \cdot t,$$

$$\therefore\ dm = \frac{2\pi}{P} \cdot dt.$$

$$\text{But } \tfrac{1}{2}\rho^2 dv = \text{ellipse} \cdot \frac{dt}{P} = \pi a^2 \sqrt{1-e^2} \cdot \frac{dt}{P};$$

$$\therefore\ a^2 \cdot \sqrt{1-e^2} \cdot dm = \rho^2 dv$$

$$= \frac{a^2 \cdot (1-e^2)^2 \cdot dv}{(1+e\cos v)^2};$$

$$\therefore\ dm = \frac{(1-e^2)^{\frac{3}{2}} \cdot dv}{(1+e\cos v)^2};$$

$$\therefore\ dm \cdot (1+e\cos v)^2 = (1-e^2)^{\frac{3}{2}} \cdot dv;$$

$$\therefore\ dm \cdot \left\{1 + e \cdot \frac{1-\tan^2\frac{1}{2}v}{1+\tan^2\frac{1}{2}v}\right\}^2 = (1-e^2)^{\frac{3}{2}} \cdot dv$$

$$= 2(1-e^2)^{\frac{3}{2}} \cdot \frac{d \cdot \tan\frac{1}{2}v}{1+\tan^2\frac{1}{2}v};$$

$$\therefore\ dm \cdot (1+e+\overline{1-e}\tan^2\tfrac{1}{2}v)^2 = 2 \cdot (1-e^2)^{\frac{3}{2}} \cdot d\tan\tfrac{1}{2}v \cdot (1+\tan^2\tfrac{1}{2}v).$$

Assume $(1-e) \cdot \tan^2\frac{1}{2}v = (1+e) \cdot \tan^2\frac{1}{2}u$;

$$\therefore\ dm \cdot (1+e)^2 \cdot (1+\tan^2\tfrac{1}{2}u)^2$$

$$= 2(1-e^2)^{\frac{3}{2}} \cdot \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \cdot d\tan\tfrac{1}{2}u \cdot \left(1 + \frac{1+e}{1-e} \cdot \tan^2\tfrac{1}{2}u\right);$$

$$\therefore dm.(1+\tan^2\tfrac{1}{2}u)^2=2d\tan\tfrac{1}{2}u.\{1-e+\overline{1+e}.\tan^2\tfrac{1}{2}u\};$$

$$\therefore dm=\frac{2.d\tan\frac{1}{2}u}{1+\tan^2\frac{1}{2}u}.\frac{1-e+\overline{1+e}.\tan^2\frac{1}{2}u}{1+\tan^2\frac{1}{2}u}$$

$$=2.d\tfrac{1}{2}u.\{\overline{1-e}\cos^2\tfrac{1}{2}u+\overline{1+e}.\sin^2\tfrac{1}{2}u\}$$

$$=du..\{1-e\cos u\};$$

$$\therefore m=u-e\sin u.$$

The elimination of u between this equation and the equation

$$\tan\tfrac{1}{2}v=\sqrt{\frac{1+e}{1-e}}.\tan\tfrac{1}{2}u,$$

will give the relation between m and v.

236. The relation between the true and the mean anomaly may be also deduced simply from geometrical considerations.

Let P (fig. 48.) be the Earth, S the Sun, APB the ellipse described, C its center, PN an ordinate meeting the circles described on AB in Q; join SQ, SP, CQ; let BSP, BCM the true and mean anomalies be measured from the *aphelion* B, and let $BCQ=u$, which angle is called the eccentric anomaly.

Then if $CN=x$, and t be the time from aphelion to P,

$$m=t.\frac{2\pi}{P}=\frac{BSP}{\text{ellipse}}.2\pi$$

$$=\frac{BSQ}{\pi a^2}.2\pi.$$

$$\text{But } BSQ=BCQ+SCQ$$

$$=\tfrac{1}{2}a^2u+\tfrac{1}{2}ae.a\sin u;$$

$$\therefore m=u+e\sin u.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

$$\text{Again } \rho^2=\overline{ae+x}|^2+y^2=\overline{ae+x}|^2+\frac{b^2}{a^2}.QN^2$$

$$=\overline{ae+x}|^2+a^2.\overline{1-e^2}|.\sin^2u$$

$$= (ae + a\cos u)^2 + a^2 . \overline{1-e^2} . \sin^2 u$$

$$= a^2 . (1 + 2e\cos u + e^2\cos^2 u);$$

$$\therefore \rho = a . (1 + e\cos u);$$

$$\therefore \cos v = \frac{ae + x}{\rho}$$

$$= \frac{ae + a\cos u}{a . (1 + e\cos u)}$$

$$= \frac{e + \cos u}{1 + e\cos u};$$

$$\therefore \frac{1 - \cos v}{1 + \cos v} = \frac{1 - e - \overline{1-e} . \cos u}{1 + e + \overline{1+e} . \cos u},$$

$$\text{or } \tan^2 \tfrac{1}{2} v = \frac{1-e}{1+e} . \tan^2 \tfrac{1}{2} u;$$

$$\therefore \tan \tfrac{1}{2} v = \sqrt{\frac{1-e}{1+e}} . \tan \tfrac{1}{2} u \ldots . (2).$$

Therefore, eliminating u between the equations (1) and (2), the relation between m and v may be found.

237. Measuring the anomalies from perihelion, or substituting for the angles u, v, m, their supplements, the preceding equations become

$$m = u - e\sin u \ldots\ldots\ldots\ldots (1).$$

$$\tan \tfrac{1}{2} v = \sqrt{\frac{1+e}{1-e}} . \tan \tfrac{1}{2} u \ldots\ldots (2).$$

The other equations in the preceding article become

$$\rho = a . (1 - e\cos u) \ldots\ldots\ldots (3).$$

$$\cos v = \frac{\cos u - e}{1 - e \cos u} \ldots\ldots\ldots\ldots (4).$$

and therefore $\cos u = \dfrac{\cos v + e}{1 + e \cos v} \ldots\ldots\ldots\ldots (5).$

238. *Given the mean anomaly, to find the true in orbits of small eccentricity.*

If e be small, $u - m = e \sin u$ is a small angle, and does not differ much from its sine; let, therefore, u' be an approximate value of u, such that

$$\sin (u' - m) = e \sin u';$$

$$\therefore \frac{\sin u' + \sin (u' - m)}{\sin u' - \sin (u' - m)} = \frac{1+e}{1-e},$$

$$\frac{\tan (u' - \frac{1}{2} m)}{\tan \frac{1}{2} m} = \frac{1+e}{1-e};$$

$$\therefore \tan (u' - \tfrac{1}{2} m) = \frac{1+e}{1-e} . \tan \tfrac{1}{2} m,$$

which determines u'.

Let now $u = u' + \phi$, where since $u' = u$ nearly, ϕ is very small;

$$\therefore u' + \phi - m = e \sin (u' + \phi);$$

$$= e \sin u' + e \cos u' . \phi, \text{ nearly};$$

$$\therefore \phi = \frac{m + e \sin u' - u'}{1 - e \cos u'}.$$

Having thus determined a near value of ϕ, let $u' + \phi = u''$, and let $u = u'' + \phi'$; then, as before,

$$\phi' = \frac{m + e \sin u'' - u''}{1 - e \cos u''}.$$

In the same manner we may go on approximating, till u is known to any required degree of accuracy, when v may be found from the equation

$$\tan \tfrac{1}{2} v = \sqrt{\frac{1+e}{1-e}} . \tan \tfrac{1}{2} u.$$

239. *To find the true anomaly in terms of the mean in a series ascending by powers of e.*

Since $\tan\frac{1}{2}v = \sqrt{\frac{1+e}{1-e}} \,.\, \tan\frac{1}{2}u = n \tan\frac{1}{2}u$, (by substitution);

$$\therefore \tfrac{1}{2}v = \tfrac{1}{2}u + \frac{n-1}{n+1} \,.\, \sin u + \tfrac{1}{2} \,.\, \left(\frac{n-1}{n+1}\right)^2 \,.\, \sin 2u + \&c. \text{ (Art. 62.)},$$

or, making $c = \frac{n-1}{n+1} = \frac{\sqrt{1+e}-\sqrt{1-e}}{\sqrt{1+e}+\sqrt{1-e}} = \frac{1-\sqrt{1-e^2}}{e}$.

$$v = u + 2(c \sin u + \tfrac{1}{2} c^2 \sin 2u + \tfrac{1}{3} c^3 \sin 3u + \&c. \;\ldots\ldots\; (1).$$

Now, by Lagrange's theorem, if $v = f(u)$, (f representing any function of u), and

$$u = m + e \sin u,$$

$$f(u) = f(m) + \sin m \,.\, f'(m) \,.\, e + \{\sin^2 m \,.\, f'(m)\}' \,.\, \tfrac{1}{2} e^2$$

$$+ \{\sin^3 m \,.\, f'(m)\}'' \,.\, \frac{1}{6} e^3 + \&c.;$$

$$\therefore u = m + \sin m \,.\, e + 2 \sin m \cos m \,.\, \tfrac{1}{2} e^2 + \&c.$$

$$\sin u = \sin m + \sin m \cos m \,.\, e + \&c.$$

$$\sin 2u = \sin 2m + \&c.$$

$$\text{Also } \sqrt{1-e^2} = 1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 + \&c.;$$

$$\therefore c = \tfrac{1}{2} e + \tfrac{1}{8} e^3 + \&c.$$

Substituting these values of u, $\sin u$, &c. in equation (1), neglecting the cube and higher powers of e, we have

$$v = m + \sin m \,.\, e + \sin 2m \,.\, \tfrac{1}{2} e^2 + \&c.$$

$$+ \sin m \,.\, e + \sin m \cos m \,.\, e^2 + \&c.$$

$$+ \sin 2m \,.\, \tfrac{1}{4} e^2 + \&c.$$

$$= m + 2 \sin m \,.\, e + \frac{5}{4} \,.\, \sin 2m \,.\, e^2 + \&c. \ldots (2).$$

The approximation has been carried on as far as e^3, merely to shew the method, which it is obvious may be extended to any higher power in the same manner. See Delambre, *Abrégé d'Astronomie,* p. 156, where the expansion is carried on as far as e^{12}.

240. If, in Lagrange's theorem, we make

$$f(u) = u + 2(c \sin u + \tfrac{1}{2}c^2 \sin 2u + \&c.);$$

$$\therefore f(m) = m + 2(c \sin m + \tfrac{1}{2}c^2 \sin 2m + \&c.),$$

$$\text{and } f'(m) = 1 + 2(c \cos m + c^2 \cos 2m + \&c.)$$

therefore denoting this quantity by M,

$$v = m + 2 . (c \sin m + \tfrac{1}{2}c^2 \sin 2m + \&c.)$$

$$+ M \sin m . e + (M \sin^2 m)' . \tfrac{1}{2}e^2 + (M \sin^3 m)'' . \frac{1}{6}e^3 + \&c.$$

where the law of the expansion is evident.

241. *To expand the mean anomaly in terms of the true.*

$$\text{Because } \cos u = \frac{e + \cos v}{1 + e \cos v};$$

$$\therefore \sin u = \sqrt{1 - \cos^2 u} = \sqrt{1 - e^2} . \frac{\sin v}{1 + e \cos v}.$$

$$\text{Also, since } c = \frac{1 - \sqrt{1 - e^2}}{e};$$

$$\therefore e = \frac{2c}{1 + c^2}, \text{ and } \sqrt{1 - e^2} = \frac{1 - c^2}{1 + c^2};$$

$$\therefore \sin u = \frac{1 - c^2}{1 + c^2} . \frac{\sin v}{1 + \frac{2c}{1 + c^2} . \cos v}$$

$$= (1 - c^2) . \frac{\sin v}{1 + c^2 + 2c \cos v}.$$

Let $2 \cos v = x + x^{-1}$;

$$\therefore \sin u = \frac{1-c^2}{2\sqrt{-1}} \cdot \frac{(x-x^{-1})}{1+c^2+c.(x+x^{-1})}$$

$$= \frac{1-c^2}{2\sqrt{-1}} \cdot \left\{ \frac{x}{1+cx} - \frac{x^{-1}}{1+cx^{-1}} \right\}$$

$$= \frac{1-c^2}{2\sqrt{-1}} \cdot \left\{ \begin{array}{c} x - cx^2 + c^2x^3 - \&c. \\ -x^{-1} + cx^{-2} - c^2x^{-3} + \&c. \end{array} \right\}$$

$$= \overline{1-c^2} \{ \sin v - c \sin 2v + c^2 \sin 3v - \&c \}.$$

Now if $\tan \frac{1}{2} u = n \tan \frac{1}{2} v$,

$$\tfrac{1}{2} u = \tfrac{1}{2} v + \frac{n-1}{n+1} \cdot \sin v + \tfrac{1}{2} \cdot \left(\frac{n-1}{n+1} \right)^2 \cdot \sin 2v + \&c.$$

But in this case, since $\tan \frac{1}{2} u = \sqrt{\frac{1-e}{1+e}} \cdot \tan \frac{1}{2} v$,

$$\frac{n-1}{n+1} = \frac{\sqrt{1-e} - \sqrt{1+e}}{\sqrt{1-e} + \sqrt{1+e}} = -c;$$

$$\therefore u = v - 2c \sin v + 2 \cdot \frac{c^2}{2} \sin 2v - \&c.$$

Therefore, substituting the values of u and $\sin u$ in the equation

$$m = u - e \sin u,$$

$$m = v - (2c + e \cdot \overline{1-c^2}) \cdot \sin v + (c^2 + ec \cdot \overline{1-c^2}) \cdot \sin 2v - \&c.$$

Def. *The equation of the center* is the difference between the true and mean anomalies, and may be obtained from Art. 240. in terms of m, and from Art. 241. in terms of v.

242. *To find the greatest equation of the center.*

When the equation of the center is a maximum,

$$d.(m-v) = 0, \text{ or } dm = dv,$$

$$\text{but } dm = \frac{\overline{1-e^2}|^{\frac{3}{2}} dv}{(1 + e \cos v)^2}$$

$$\therefore\ (1+e\cos v)^2 = \overline{1-e^2}|^{\frac{3}{2}};$$

$$\therefore\ 1+e\cos v = \overline{1-e^2}|^{\frac{3}{4}};$$

$$\therefore\ \cos v = \frac{1-\overline{1-e^2}|^{\frac{3}{4}}}{e}.$$

$$\text{Again, } \rho = a.(1-e\cos u),$$

$$\text{or } \frac{a.\overline{1-e^2}}{(1+e\cos v)} = a.(1-e\cos u);$$

$$\therefore\ \overline{1-e^2}|^{\frac{1}{4}} = 1-e\cos u,$$

$$\text{since } 1+e\cos v = \overline{1-e^2}|^{\frac{3}{4}};$$

$$\therefore\ \cos u = \frac{1-\overline{1-e^2}|^{\frac{1}{4}}}{e}.$$

From these equations v and u may be found, and then m from the equation

$$m = u - e\sin u.$$

The difference of the values of m and v thus found, is the greatest equation of the center.

243. *Second method of finding the greatest equation of the center.*

$$\text{Because } \tan\tfrac{1}{2}v = \sqrt{\frac{1+e}{1-e}}.\tan\tfrac{1}{2}u;$$

$$\therefore\ \tan\tfrac{1}{2}v - \tan\tfrac{1}{2}u = \tan\tfrac{1}{2}u\left\{\sqrt{\frac{1+e}{1-e}}-1\right\};$$

multiplying each side by $\cos\frac{1}{2}v.\cos\frac{1}{2}u$;

$$\sin\tfrac{1}{2}(v-u) = \cos\tfrac{1}{2}v.\sin\tfrac{1}{2}u.\left\{\sqrt{\frac{1+e}{1-e}}-1\right\},$$

$$\therefore\ \sin\tfrac{1}{2}(v-u) = \frac{\sin u}{2}.\frac{\cos\frac{1}{2}v}{\cos\frac{1}{2}v}.\left\{\sqrt{\frac{1+e}{1-e}}-1\right\}.$$

But $\cos^2 \frac{1}{2} v = \frac{1}{2} . (1 + \cos v)$

$$= \frac{1}{2} . \left\{ 1 + \frac{\cos u - e}{1 - e \cos u} \right\}$$

$$= \frac{1}{2} . \frac{1 - e + \overline{1 - e} . \cos u}{1 - e \cos u}$$

$$= \frac{\overline{1 - e} . \cos^2 \frac{1}{2} u}{1 - e \cos u};$$

$$\therefore \cos \frac{1}{2} v = \frac{\sqrt{1 - e} . \cos \frac{1}{2} u}{(1 - e \cos u)^{\frac{1}{2}}};$$

$$\therefore \sin \frac{1}{2}(v - u) = \frac{\sin u}{2} \left\{ \frac{\sqrt{1 + e} - \sqrt{1 - e}}{(1 - e \cos u)^{\frac{1}{2}}} \right\}.$$

Let $e = \sin \epsilon$.

Then, when the equation of the center is a maximum,

$$1 - e \cos u = \cos^{\frac{1}{2}} \epsilon, \text{ (Art. 242)},$$

and $\sqrt{1 + \sin \epsilon} - \sqrt{1 - \sin \epsilon} = 2 \sin \frac{1}{2} \epsilon$,

(by the formula of verification);

$$\therefore \sin \frac{1}{2}(v - u) = \sin u . \frac{\sin \frac{1}{2} \epsilon}{\cos^{\frac{1}{4}} \epsilon};$$

$$\therefore v - u = 2 \sin^{-1} . \frac{\sin u . \sin \frac{1}{2} \epsilon}{\sqrt[4]{\cos \epsilon}},$$

and $u - m = e \sin u$;

$$\therefore v - m = 2 \sin^{-1} . \left(\frac{\sin u . \sin \frac{1}{2} \epsilon}{\sqrt[4]{\cos \epsilon}} \right) + e \sin u.$$

This expression for the greatest equation of the center may, if requisite, be expanded in powers of e, substituting for $\cos u$ its value

$$\frac{1 - \sqrt[4]{1 - e^2}}{e}. \text{ (Art. 242.)}$$

244. In orbits of small eccentricity, $v = m + 2e \sin m$, nearly;

$$\therefore v - m = 2e \sin m, \quad \text{(Art. 239.)}$$

the maximum value of which is $2e$; therefore in orbits of small eccentricity, the greatest equation of the center equal twice the eccentricity.

If the anomalies are measured from the apogee, the greatest equation of the center is the greatest value of $m - v$, which $= -$ the greatest value of $v - m$; therefore the greatest equation of the center must change its sign, when the anomalies are measured from the apogee; that is, when e is changed into $-e$; hence the greatest equation of the center can contain only the odd powers of e, and the preceding result which is obtained by neglecting e^2, is the same as would have been found by neglecting e^3, and higher powers.

245. *Given the time of the year, to find the Sun's longitude, and conversely, given the longitude, to find the time.*

Let T be the time, t the time of the Sun's passing the perigee, s the Sun's mean daily motion; then $s.(T-t)$ is the Sun's mean anomaly, and if to this be added the equation of the center E, we get $s(T-t)+E$ for the true anomaly; and therefore $s.(T-t)+E+p$ for the Sun's true longitude l, p being the longitude of the perigee. If the longitude l be given, to find the time T, we have given $l-p=$ the Sun's true anomaly, from which, subtracting the equation of the center, we know $l-p-E$ the mean anomaly, and therefore

$$s.(T-t) = l - p - E;$$

$$\therefore T = t + \frac{l-p-E}{s}.$$

246. As the planets revolve round the Sun in the same manner as the Earth in orbits of small eccentricity, the preceding equations may be applied to the determination of the planetary motions.

If m be the mean anomaly in a planet's orbit corresponding to a time t from perihelion, we have, as for the Earth,

$$m = \frac{360}{P} \cdot t.$$

But if a be the mean distance of the planet,

$$P : 1 \text{ year } (1) :: a^{\frac{3}{2}} : 1^{\frac{3}{2}};$$

$$\therefore m = \frac{360}{a^{\frac{3}{2}}} \cdot t,$$

t being expressed in parts of a year, and a in parts of the Earth's mean distance.

The orbits described by comets are very eccentric ellipses, which, with few exceptions, may be considered as parabolas for those parts during the description of which the comet is visible from the Earth. The consideration of the circumstances of a body's motion in a very eccentric ellipse, is not sufficiently elementary for the present work. The two following propositions will, however, be of use in the next chapter.

247. *To find the radius vector, and the time corresponding to a given true anomaly, in an orbit where $e = 1$, nearly.*

Assuming as before,

$$\rho = \frac{a \cdot \overline{1 - e^2}}{1 + e \cos v};$$

$$\text{But } 1 + e \cos v = 1 + e \cdot \frac{1 - \tan^2 \frac{1}{2} v}{1 + \tan^2 \frac{1}{2} v}$$

$$= \frac{1 + e + \overline{1 - e} \cdot \tan^2 \frac{1}{2} v}{\sec^2 \frac{1}{2} v}$$

$$= \cos^2 \tfrac{1}{2} v \cdot \{1 + e + \overline{1 + e} \cdot \tan^2 \tfrac{1}{2} u\};$$

$$\therefore \rho = \frac{a \cdot \overline{1 - e}}{\cos^2 \frac{1}{2} v \cdot (1 + \tan^2 \frac{1}{2} u)} = a \cdot \overline{1 - e} \cdot \frac{\cos^2 \frac{1}{2} u}{\cos^2 \frac{1}{2} v}$$

$$= p \cdot \frac{\cos^2 \frac{1}{2} u}{\cos^2 \frac{1}{2} v} \dots (1),$$

p being the perihelion distance $a - ae$.

$$\text{Again } dt = \frac{a^{\frac{3}{2}} dm}{2\pi}, \text{ Art. 246.}$$

$$= \frac{a^{\frac{3}{2}} . (1 - e^2)^{\frac{3}{2}} . dv}{2\pi (1 + e \cos v)^2} . \text{ Art. 245.}$$

Let $x = 1 - e$, where, since e is nearly equal to 1, x is very small;

$$\therefore x = \frac{a - ae}{a} = \frac{p}{a}, \text{ and } a = \frac{p}{x};$$

$$\therefore dt = \left(\frac{p}{x}\right)^{\frac{3}{2}} . \frac{(2x - x^2)^{\frac{3}{2}}}{\pi} . \frac{d . \frac{1}{2} v}{(1 + e \cos v)^2},$$

$$\text{but } d \tfrac{1}{2} v = \frac{d \tan \frac{1}{2} v}{1 + \tan^2 \frac{1}{2} v}.$$

$$= \frac{dy}{1 + y^2}, \text{ (making } y = \tan \tfrac{1}{2} v),$$

$$\text{and } 1 + e \cos v = 1 + \overline{1 - x} . \frac{1 - y^2}{1 + y^2}$$

$$= \frac{2 - x . \overline{1 - y^2}}{1 + y^2};$$

$$\therefore dt = \frac{p^{\frac{3}{2}}}{\pi} . (2 - x)^{\frac{3}{2}} . \frac{dy . (1 + y^2)}{(2 - x . \overline{1 - y^2})^2}$$

$$= \frac{p^{\frac{3}{2}}}{\pi \sqrt{2}} . (1 - \tfrac{1}{2} x)^{\frac{3}{2}} . \frac{dy . (1 + y^2)}{\{1 - \frac{1}{2} x . (1 - y^2)\}^2}$$

$$= \frac{p^{\frac{3}{2}}}{\pi \sqrt{2}} . (1 - \tfrac{1}{2} x)^{\frac{3}{2}} . dy . (1 + y^2) . \left\{1 + x . \overline{1 + y^2} + 3x^2 . \frac{\overline{1 - y^2}|^2}{4} + \&\text{c.}\right\}$$

$$= \frac{p^{\frac{3}{2}}}{\pi \sqrt{2}} \{1 - \tfrac{3}{4} x\} . dy . \{(1 + y^2) + x . (1 - y^4)\},$$

neglecting x^2, x^3, &c.

$$\therefore t = \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}} . (1 - \tfrac{3}{4}x) . \left\{ y + \frac{y^3}{3} + x . \left(y - \frac{y^5}{5} \right) \right\}$$

$$= \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}} \left\{ y + \frac{y^3}{3} + \left(\tfrac{1}{4} y - \frac{y^3}{4} - \frac{y^5}{5} \right) x \right\}$$

$$= \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}} \{ \tan \tfrac{1}{2} v + \tfrac{1}{3} \tan^3 \tfrac{1}{2} v$$

$$+ (\tfrac{1}{4} . \tan \tfrac{1}{2} v - \tfrac{1}{4} . \tan^3 \tfrac{1}{2} v + \tfrac{1}{5} \tan^5 \tfrac{1}{2} v) x \} \; \ldots (2).$$

248. Cor. When x is 0, the curve described becomes a parabola, and the equations (1) and (2) become

$$\rho = \frac{p}{\cos^2 \frac{1}{2} v},$$

$$t = \frac{p^{\frac{3}{2}}}{\pi\sqrt{2}} . \{ \tan \tfrac{1}{2} v + \tfrac{1}{3} \tan^3 \tfrac{1}{2} v \},$$

which may be applied to the determination of the motions of comets, since very few of their paths differ sensibly from parabolas, whilst they are near the Earth's orbit.

249. *To find the time of describing a given parabolic arc.*

Let QR (fig. 49.) be the arc of a parabola, Pv (x) a diameter bisecting the chord QR in v. Join SP (y), SQ, SR, and draw Pp, Qq, Rr perpendicular to the directrix qr. Then, by the property of the parabola, $Px = Pv$, $Qq = SQ$, $Rr = SR$; therefore,

$$y + x = pv = \tfrac{1}{2} . (Qq + Rr) = \tfrac{1}{2} . (SQ + SR),$$

$$\text{and } 2\sqrt{yx} = Qv \qquad = \tfrac{1}{2} . QR \ldots\ldots (1);$$

$$\therefore y \pm 2\sqrt{yx} + x = \tfrac{1}{2} . (SQ + SR \pm QR).$$

Let $SQ + SR + QR = 2\alpha^2$, $SQ + SR - QR = 2\beta^2$;

$$\therefore \sqrt{y} + \sqrt{x} = \alpha,$$

$$\sqrt{y}-\sqrt{x}=\ \beta;$$

$$\therefore\ y \quad =\tfrac{1}{4}\,.\,(\alpha+\beta)^2,\ x=\tfrac{1}{4}\,.\,(\alpha-\beta)^2\ldots(2).$$

Let now t be the time from Q to R, a the area described round S in the unit of time, v the velocity at P, and θ the angle SxQ, which is the same as that between SP and the tangent at P; then

$$SQPR=\triangle SQR+QPR$$

$$=\tfrac{1}{2}\,QR\,.\,(y-x)\sin\theta+\tfrac{2}{3}\,QR\,.\,x\sin\theta\ldots(2).$$

But, if m be the force at a distance unity from S,

$$v=\sqrt{\frac{2m}{SP}};$$

$$\therefore\ a=\tfrac{1}{2}\sqrt{\frac{2m}{SP}}\,.\,SP\sin\theta=\tfrac{1}{2}\sqrt{2my}\,.\sin\theta$$

$$=\tfrac{1}{8}\sqrt{\frac{2m}{x}}\,.\,QR\,.\sin\theta\ \text{ from (1).}$$

Therefore, dividing $SPQR$ by a, we get

$$t=\frac{8}{\sqrt{2m}}\,.\,\{\tfrac{1}{2}(y-x)+\tfrac{2}{3}\,x\}\,.\sqrt{x}$$

$$=\frac{8}{\sqrt{2m}}\,\{\tfrac{1}{2}\alpha\beta+\tfrac{1}{6}\,.\,(\alpha-\beta)^2\}\,.\,\tfrac{1}{2}(\alpha-\beta)$$

$$=\frac{2}{3\sqrt{2m}}\,\{\alpha^2+\alpha\beta+\beta^2\}\,.\,(\alpha-\beta)$$

$$=\frac{2}{3\sqrt{2m}}\,.\,\{\alpha^3-\beta^3\}$$

$$=\frac{1}{6\sqrt{m}}\,\{(SQ+QR+SR)^{\frac{3}{2}}-(SQ+SR-QR)^{\frac{3}{2}}\}.$$

If a day be taken for the unit of time, and the radius of the Earth's orbit for the unit of distance;

$$365.25 = \frac{\oplus \text{ orbit}}{\frac{1}{2} \cdot 1 \cdot \text{vel. of } \oplus} = \frac{\pi}{\frac{1}{2} \cdot \sqrt{m}};$$

$$\therefore \frac{1}{6\sqrt{m}} = \frac{365}{12\pi},$$

and the time of describing QR

$$= \frac{365}{12\pi} \cdot \{(SQ + SR + QR)^{\frac{3}{2}} - (SQ + SR - QR)^{\frac{3}{2}}.$$

CHAP. XIII.

THE EQUATION OF TIME.

250. It has been before observed that a fictitious Sun describing the equator with the Sun's mean motion in longitude would, by its presence on the meridian, mark the mean time of 12 o'clock, and its hour angle at any other instant converted into time would be the corresponding mean time. In the same manner, the apparent time of noon, and the apparent hour at any other instant, may be deduced from the presence on the meridian, and from the hour angle of the real Sun. The difference of the hour angles of the true and mean Sun, or, which is the same thing, the difference of their right ascensions when converted into time, is the equation of time expressed in parts of mean time.

251. *To find the equation of time.*

Let l be the longitude, a the right ascension, and L the mean longitude of the Sun, or the right ascension of the fictitious Sun, all measured from the mean place of the equinox unaffected by nutation, l', a', L' the same quantities measured from the true place of the equinox; and let r be the reduction of the Sun's longitude to the equator (Art. 79.), or let $l + r = a$.

$$\text{Then } a' = l + r - 18''.03 \cos\omega \,.\, \sin ☽☊, \quad \text{(Art. 196.)}$$

$$L' = L - 18''.03 \,.\, \sin ☽☊;$$

therefore, the equation of time

$$= \frac{1}{15} \cdot (a' - L')$$

$$= \frac{1}{15} \cdot \{l - L + r + 18''.03 \,.\, \sin \text{☽☊} \cdot (1 - \cos \omega)\},$$

To this must be added a small quantity $\frac{1}{15} \cdot P$, arising from the disturbing forces of the planets, on the Sun's longitude, the calculation of which depends on Physical Astronomy; therefore, the whole equation of time

$$= \frac{1}{15} \cdot (l - L + r - 18''.03 \,.\, \sin \text{☽☊} \,.\, \overline{1 - \cos \omega}) + \frac{1}{15} \cdot P,$$

If $\omega = 23^0\, 28'$, the equation of time

$$= \frac{1}{15} \cdot (l - L + r + P) + 0''.099 \sin \text{☽☊}.$$

252. The true longitude may be obtained in terms of the mean, or conversely, by the equations of the preceding Chapter, and thus the equation of time may be determined in terms of the true or mean longitude of the Sun; Littrow, (Vol. II. p. 71.) has given expressions in which the equation of time is expressed in a series of sines and cosines of multiple arcs of the true, and also of the mean longitude of the Sun, from which tables may be constructed *.

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* If E be the equation of time, calculated upon the supposition that the eccentricity of the Earth's orbit $= .016791$, the longitude of the perihelion $(p) = 279^\circ\, 29'\, 33''$, and the obliquity $= 23^\circ\, 27'\, 57''$, which values are nearly sufficient from the year 1800 to 1900,

$$\begin{aligned} E = {} & 461''.786 \;\sin (l - p) - 593.146 \;\sin 2\,l \\ & - 2.907 \;\sin 2\,(l - p) + 12.793 \;\sin 4\,l \\ & + 0.022 \;\sin 3\,(l - p) - 0.368 \;\sin 6\,l \\ & + 0.099 \;\sin \text{☽☊} + \frac{1}{15} P. \end{aligned}$$

Or

253. The equation of time in the Nautical Almanack is given only for the time when the Sun is on the meridian of Greenwich, and for any other time, or when the Sun is on any other meridian, the proportional part must be taken. Let E and E' be the values of the equation of time on two successive noons, taken from the Nautical Almanack, $E+\Delta E$ the equation of time at some intermediate instant, when the Sun is on a meridian h^0 west of Greenwich. Then, the increment of E may be considered proportional to the time, or to the hour angle described by the Sun nearly;

$$\therefore\ 360^0 : h^0 :: E'-E : \Delta E = \frac{h^0}{360^0}.(E'-E);$$

$$\therefore\ E+\Delta E = E + \frac{h^0}{360^0}.(E'-E).$$

254. From the expression for the equation of time (Art. 251.), it appears that if the Sun moved uniformly in the ecliptic, or $l=L$, the equation of time would still vary on account of the variable reduction r, and if the ecliptic coincided with the equator, in which case r would equal 0, the equation of time would change from changes in the quantity $l-L$; therefore, the equation of time may be considered as arising from two causes; the obliquity of the ecliptic, and the Sun's unequal motion in longitude. The effects of these may be considered separately, by conceiving a star to move uniformly in the ecliptic with the Sun's mean motion in longitude; then, the difference

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Or $$\begin{aligned} E = {} & 79''.36 \sin L - 597.08 \sin 2L - 3.42 \sin 3L \\ & + 13.25 \sin 4L + 0.15 \sin 5L - 0.40 \sin 6L \\ & + 435.8 \cos L + 1.6 \cos 2L - 18.8 \cos 3L \\ & - 0.2 \cos 4L + 0.9 \cos 5L \\ & + 0.1 \sin \text{☽☊} - 0.1 \sin(2L + \text{☽☊}) + \frac{1}{15} P. \end{aligned}$$

of the times of transit of this star and the Sun over the meridian, would be the equation of time caused by the Sun's unequal motion in the ecliptic, and the difference of the times of transit of the star, and the fictitious Sun in the equator, would be the equation of time caused by the obliquity.

255. *To compare the Sun's motion in longitude and right ascension.*

Let ω, l, a, δ be, as before, the obliquity, longitude, right ascension and declination of the Sun.

$$\text{Then } \cos\omega = \tan a \,.\, \cot l,$$

$$\text{Or } \tan a = \cos\omega \,.\, \tan l;$$

$$\therefore \frac{da}{\cos^2 a} = \frac{\cos\omega \,.\, dl}{\cos^2 l} = \frac{\cos\omega \,.\, dl}{\cos^2 a \,.\, \cos^2\delta};$$

$$\therefore\ da = dl \,.\, \cos\omega \,.\, \text{sect}^2\,\delta.$$

256. *To find when the equation of time caused by the obliquity is additive or subtractive.*

Let ♈S (fig. 50.) be the ecliptic, ♈D the equator, SD a circle of declination passing through the Sun S, here supposed to move uniformly in the ecliptic; take ♈M = ♈S, then, DM converted into time is the equation of time caused by the obliquity.

Now cos ♈SD = sin ♈ . cos ♈D, which is positive or negative, according as ♈D is less or greater than 90°; therefore, ♈SD is an acute angle when ♈D is less, and obtuse when ♈D is greater than 90°; but ♈DS is always =90°; therefore, since the greater side subtends the greater angle, ♈S or ♈M is greater than ♈D when ♈D is less than 90°, or from an equinox to a solstice, and ♈M is less than ♈D when ♈D is greater than 90°, or from a solstice to an equinox. At the solstice ♈S and ♈D both equal 90°, and M coincides with D. M also coincides with D at the next equinox, since then ♈S and ♈D both equal 180°. The same effects must take place in the other half of the orbit; therefore, as the Earth revolves round its axis, a given meridian will pass through S before M, from an equinox to a solstice, and after M, from a solstice to an equinox; that is, the equation of time caused by the obliquity is subtractive from an equinox to a solstice, 0 at the solstices and equinoxes, and additive from a solstice to an equinox.

257. Cor. Since the equation of time, caused by the obliquity, is 0 at the equinox and solstice, it must be a maximum at some intermediate point. At this point $d.(l-a) = 0$, or $dl = da$;

$$\therefore \ 1 = \sec^2 \delta . \cos \omega;$$

$$\therefore \ \cos \delta = \sqrt{\cos \omega}.$$

From this equation δ may be found, and from it l and a, by the equations

$$\cos \delta = \sin \omega . \sin l,$$

$$\sin a = \tan \delta . \cot \omega.$$

The numerical values of these quantities are

$$l = 46^0 \ 14' \ 0'',$$

$$a = 43 \ 43 \ 50;$$

therefore, $l-a$, or the equation of time expressed in degrees

$$= 2^0 \ 20' \ 10''.$$

258. *To find when the equation of time, caused by the unequal motion in the ecliptic, is additive or subtractive.*

When the Sun is in the perigee, the mean anomaly coincides with the true; but as the Sun is then moving fastest, the motion is greater than the mean motion, and the true place of the Sun precedes the mean till the apogee, when the mean and true places of the Sun coincide. After passing the apogee, the Sun's motion being slower than the mean, the mean place precedes the true till perigee, when they again coincide.

Therefore, the equation of time, caused by the Sun's unequal motion in the ecliptic, is additive from perigee to apogee, subtractive from apogee to perigee, and 0 at those points.

259. The greatest separation of the Sun's mean and true places in the ecliptic, or the greatest equation of the center, is $1^0\ 55'\ 33''$; which will produce the greatest separation of their places referred to the equator, when this quantity, substituted in the equation of Art. 255, for dl, produces the greatest value of da. In this case,

$$da = (1^0\ 55'\ 33'') \cos\omega \,.\, \text{sect}^2\,\delta,$$

which is greatest when $\delta = \omega = 23^\circ\ 28'$; therefore, making $\delta = 23^0\ 28'$,

$$da = 2^0\ 5'\ 55'',$$

which, expressed in degrees, is the greatest equation of time that could be caused by the Sun's unequal motion in the ecliptic.

260. *The equation of time is 0 four times a year.*

Let $A\Upsilon S$ (fig. 51.) be the ellipse described by the Sun round the Earth, p and a the perigee and apogee, A and Υ the autumnal and vernal equinoxes, S and W the summer and winter solstices, and M the place about 46^0 degrees from Υ, where the equation of time, caused by the obliquity, is a maximum.

Let E be the whole equation of time, t and t' the separate parts caused by the obliquity, and the Sun's elliptical motion respectively, then, the maximum value of t is about $2^\circ\ 20'$, and greater than that of t'. Now,

$$t \text{ is } - \text{ through } AW,\ \Upsilon S;\ + \text{through } W\Upsilon,\ SaA;\ \text{and } 0 \text{ at } \begin{cases} A \\ W \\ \Upsilon \\ S \end{cases}$$

$$t' \ldots\ldots\ldots\ldots aWp;\quad \ldots\ldots\ldots\ldots p\Upsilon a;\ 0 \text{ at } \begin{cases} p \\ a \end{cases}$$

$$\therefore \text{ from } A \text{ to } W,\ E = -t - t', \text{ and at } W,\ E = -t',$$

$$\text{from } W \text{ to } p,\ E = t - t', \ \ldots\ldots\ldots p,\ E = t;$$

therefore, since E was negative at W and positive at p, there is some point between W and p, at which $E = 0$.

Again,

$$\text{from } p \text{ to } \Upsilon,\ E = +t + t', \text{ and at } \Upsilon,\ E = +t',$$

$$\text{from } \Upsilon \text{ to } S,\ E = -t + t', \text{ and at } S,\ E = +t'.$$

But since at M t has its greatest value, which is greater than t'; therefore, at M E is negative, and it is positive at Υ and S; therefore, at one point between Υ and M, and another between M and S, $E = 0$,

$$\text{from } S \text{ to } a,\ E = +t + t', \text{ and at } a,\ E = +t,$$

$$\text{from } a \text{ to } A,\ E = +t - t', \text{ and at } A,\ E = -t',$$

therefore, at some point between a and A, E must vanish for the fourth time.

261. The equation of time varies continually in different years, on account of the change in the position of the line of apsides ap, which makes now an angle of about 9^0 with the line of solstices SW. If these lines coincided, the equation of time would be 0 at the solstices, since the separate parts of the equation t and t' would each = 0 *.

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* The line of apsides ap, not coinciding with the line of solstices SW, causes also the lengths of the seasons to differ. The angles ΥES, SEA, AEW, $WE\Upsilon$, which are described in the successive seasons of Spring, Summer, Autumn and Winter, are all equal; but it is obvious from the parts of the ellipse which they subtend, that the area $SE\Upsilon$ is less than SEA, and greater than $WE\Upsilon$, or AEW, and that AEW is greater than

$WE\Upsilon$

WE♈; therefore, since equal areas are described in equal times round *E*, the seasons in the order of their duration are winter, autumn, spring and summer, winter being the shortest and summer the longest. The difference between these two, however, is not four days, and of course, the difference between any two others is less. If *ap* coincided with *SW*, the spring would equal the summer, and the autumn the winter.

◆

CHAP. XIV.

ON THE PLANETS.

262. The Planets, as seen from the Earth, move sometimes from west to east, and at other times are stationary, or move from east to west. These appearances, which prove that they do not revolve round the Earth, can all be explained upon the hypothesis of their describing nearly circular orbits round the Sun, subject to the same laws of motion as the Earth.

263. There are ten planets now known, which are denoted by their appropriate symbols, Mercury (☿), Venus (♀), Mars (♂), Vesta (⚶), Juno (⚵), Ceres (⚳), Pallas (⚴), Jupiter (♃), Saturn (♄), Uranus (♅): of these Mercury, Venus, Mars, Jupiter and Saturn are most conspicuous, and have been observed from the most ancient times; the others, of which Uranus is the chief, are amongst the greatest astronomical discoveries of the last forty years. The Earth, considered as a planet, is denoted by the symbol ⊕, and its orbit lies between those of Venus. and Mars. With few exceptions the orbits of the planets are nearly circular, and inclined at small angles to the ecliptic.

264. The angle which the distance from the Sun of any planet subtends at the Earth, is called the *Elongation* of the planet. The greatest elongation of Venus is 45°, and that of Mercury still less; and they are sometimes seen between the Earth and Sun, or as dark spots moving over the Sun's disk; from which it is evident, that their orbits round the Sun lie within that of the Earth. The elongations of the other planets have all degrees of magnitude, from 0° to 180°, and they are

never seen between the Earth and Sun; therefore their orbits must lie without that of the Earth. From these circumstances Mercury and Venus are called *inferior* planets, and the others *superior*. The periods round the Sun of the inferior planets are less, and the periods of the superior planets all greater, than a year.

265. The places of the planets as seen from the Earth are called their *Geocentric* places, and are determined by their geocentric latitudes and longitudes computed from the observed right ascensions and declinations. The places as seen from the Sun are called *Heliocentric;* the *Heliocentric Latitude* of a planet being its angular distance from the ecliptic as seen from the Sun, and the heliocentric longitude being the angle subtended at the Sun between the first point of Aries, and the planet's place referred to the ecliptic.

266. The intersection of the plane of a planet's orbit with the ecliptic is called the *Line of Nodes;* that point through which the planet passes in moving from the south to the north side of the ecliptic being called the *Ascending Node* (☊), and the other the *Descending Node* (☋).

The angle at the Sun's center between the ascending node and the perihelion distance of the planet, is the *Distance of the Perihelion from the Node,* which added to the longitude of the ascending node, is the *Longitude of the Perihelion.* The angle at the Sun's center between the radius vector of the planet in its orbit, and the ascending node, is called the *Argument of Latitude.* The argument of latitude together with the longitude of the ascending node, is called the *Longitude of the Planet in its Orbit.*

If P (fig. 52.) be a planet in its orbit QPa, S the Sun, π the perihelion, Sa the line of nodes, S♈ passing through ♈ in the plane of the ecliptic ySx; then

♈ $Sa\,(n) =$ the longitude of the ascending node,

$\pi Sa +$ ♈ $Sa(p) =$ . . . . perihelion,

$PSa\,(u) =$ . . . . argument of latitude;

therefore, if π be the distance of the perihelion from the node, v the true anomaly, l' the reduced longitude on the ecliptic, and r the reduction we have from the figure,

$$p - n = \pi,$$
$$u = \pi + v = p - n + v,$$
$$l = n + u = p + v,$$
$$l' = l + r.$$

267. Before proceeding farther with the planets, it is necessary to shew how to determine the geocentric place of a planet from the heliocentric, in order to make use of the tables of the planet's motions; because all the computed motions of a planet are calculated with reference to the Sun; and how to find the heliocentric place from the geocentric, in order to make use of observations which are necessarily made at the Earth.

268. *Given the heliocentric, to find the geocentric place of a planet at any time.*

Let S (fig. 53.) be the Sun, and E the Earth in the plane of the ecliptic ySx, P a planet, PN perpendicular to the plane ySx, and NM perpendicular to Sx, $S\Upsilon$, $E\Upsilon$ parallel to each other, passing through the first point of Aries. Let λ $(NE\Upsilon)$, $\beta\,(NEP)$, $\rho\,(EP)$ be the geocentric latitude, longitude and distance of P, respectively; $l\,(NS\Upsilon)$, $b\,(NSP)$, $r\,(SP)$, the corresponding heliocentric longitude, latitude and distance of P; $L\,(ES\Upsilon)$, $R\,(SE)$ the heliocentric longitude of the Earth, and the distance of the Earth from the Sun. Let also r' and ρ', which are called the curtate distances, be the projections of r and ρ on the ecliptic. Then we have

$$r' = r\cos b,\ \ \rho' = \rho\cos\beta,\ \ NSM = l - L,\ \ NEM = \lambda - L;$$
$$\therefore\ r'\cos(l - L) - R = SM - SE = \rho'\cos(\lambda - L),$$
$$r'\sin(l - L) = \quad MN \quad = \rho'\sin(\lambda - L),$$
$$r'\tan b = \quad PN \quad = \rho'\tan\beta;$$

$$\therefore \tan(\lambda - L) = \frac{r' \sin(l - L)}{r' \cos(l - L) - R} \ldots\ldots (1),$$

$$\rho' = r' . \frac{\sin(l - L)}{\sin(\lambda - L)} \ldots\ldots\ldots (2),$$

$$\tan\beta = \frac{r' \tan b}{\rho'} \ldots\ldots\ldots\ldots (3).$$

From these three equations may be obtained successively λ, ρ' and β, since r', l and b are given, and L and R are known from the time.

269. Cor. If perpendiculars Ee, Nn, are drawn on $E♈$, $S♈$,

$$r' \cos l - R \cos L = Sn - Se = Eo = \rho' \cos\lambda,$$

$$r' \sin l - R \sin L = Nn - Ee = No = \rho' \sin\lambda.$$

270. *Given the geocentric, to find the heliocentric place of a planet.*

Let S (fig. 52.) be the Sun, E the Earth, P a planet in its orbit, Sa the line of nodes, PN perpendicular to the ecliptic ySx, and NM to SM; draw Eo parallel, and EQ perpendicular to Sa, and join PM; then PM is perpendicular to SM, and PMN is the inclination of the orbit (e), PSM the argument of latitude (u), and $MS♈$ the longitude of the node (n).

Let β, λ, ρ, ρ', &c. be as in the preceding article.

$$\text{Then, } \rho \cos\beta . \cos(\lambda - n) = EN . \cos NEo = SM - SQ$$

$$= r \cos u - R \cos(L - n) \ldots . (1),$$

$$\rho \cos\beta \sin(\lambda - n) = No = NM - EQ,$$

$$= r \sin u \cos e - R \sin(L - n) . . (2),$$

$$\rho \sin\beta = PN = r \sin u . \sin e \ldots\ldots\ldots\ldots (3).$$

Eliminating R between equation (1) and (2), we have

$$\rho \cos\beta . \{\cos(\lambda - n) . \sin(L-n) - \sin(\lambda - n) . \cos(L-n)\}$$

$$= r . \cos u . \sin(L-n) - r \sin u \cos e . \cos(L-n),$$

and dividing this by (3), it becomes

$$\cot\beta . \sin(L-\lambda) = \frac{\cot u . \sin(L-n) - \cos e . \cos(L-n)}{\sin e};$$

$$\therefore \cot u = \frac{\sin e . \cot\beta . \sin(L-\lambda) + \cos e . \cos(L-n)}{\sin(L-n)} \ldots (4).$$

Eliminating $r \sin u$ from (2) and (3), we have

$$\rho . \{\sin\beta . \cos e - \cos\beta \sin(\lambda - n) . \sin e\} = R \sin(\lambda - n) . \sin e;$$

$$\therefore \rho = \frac{R \sin(\lambda - n) . \sin e}{\sin\beta \cos e - \cos\beta \sin(\lambda - n) . \sin e} \ldots\ldots\ldots\ldots (5).$$

Having found ρ and u from equations (4) and (5),

$$r = \frac{\rho \sin\beta}{\sin u . \sin e} \text{ is determined.}$$

271. COR. 1. The heliocentric place is supposed to be determined here by the argument of latitude from which the heliocentric latitude and longitude may be determined. For making S (fig. 54.) the center of a sphere which meets the lines SP, SM, SN in p, m, n, we have, in the spherical triangle pmn,

$$pm = u, \quad nm = l - n, \quad pn = b, \quad pmn = e,$$

and the angle pnm is a right angle; therefore

$$\tan(l-n) = \cos e . \tan u,$$

$$\sin b = \sin e . \sin u,$$

which equations determine l and b.

272. Cor. 2. From the same triangle,

$$\cos u = \cos b \cdot \cos (l - n).$$

273. Cor. 3. If x, y, z be the co-ordinates of P measured from S, and SM (fig. 52.) coincide with $S♈$ or the line of equinoxes be taken for the axis of x,

$$x = Eo + SQ = \rho \cos \beta \cos \lambda + R \cos L,$$

$$y = No + EQ = \rho \cos \beta \sin \lambda + R \sin L,$$

$$z = \quad PN \quad = \rho \sin \beta.$$

274. The equations for determining the heliocentric place of a planet from the geocentric, contain the values n, e, &c. depending upon the position of the planet's orbit, of which they require the previous determination. The elements of a planet's orbit are, 1. The inclination of the orbit to the ecliptic. 2. The longitude of the ascending node. 3. The longitude of the perihelion. 4. The mean distance. 5. The eccentricity. 6. The epoch of the planet's being in the perihelion. The two first elements determine the plane of the body's motion, the third the position of the axis, the 4th and 5th the form of the orbit, and the 6th the position of the body at a given time. The period may be substituted for the mean distance, since

$$P : 1 \text{ year} :: a^{\frac{3}{2}} : 1^{\frac{3}{2}}.$$

275. From the observed geocentric right ascension and declination of a planet at any time, the geocentric latitude and longitude may be found, from which the corresponding values of u and r may be obtained in terms of the elements, by the equations of Art. 270. But the values of r and u (=the longitude of the perihelion + the true anomaly) at the time of observation, may be found in terms of the same elements, by the methods of the preceding Chapter. Therefore by equating values thus found, we obtain two equations for determining the elements. From two other observations four more equations may be obtained, so that three complete observations are sufficient for determining

the six elements of a planet's path. If the path be a circle, the eccentricity and place of the perihelion are not required, and two observations are sufficient for determining the four elements. If the path is a parabola, since the major axis is not required, the three observations furnish one equation more than necessary.

276. The above, although a theoretical, is not a practical, method of finding the elements of a planet's orbit, as the deduced equations, even for the parabola, are much too involved to admit of being solved, and of course would be much more so for the ellipse; so that the direct solution of the problem may be considered impossible. Even the indirect modes which astronomers have adopted for solving the general problem, are of very considerable difficulty, and not adapted to an elementary treatise like the present. We shall therefore confine ourselves to the cases of a circular or parabolic orbit*.

277. *To find the circular orbit of a planet from two observations.*

Let a be the radius of the orbit, x, y, z co-ordinates of the planet at the time of the first observation, the Sun being the origin, and the axis of x passing through the first point of Aries.

$$\text{Let } z+py+qx=0 \;\ldots\ldots\; (1),$$
$$z^2+x^2+y^2=a^2 \;\ldots\ldots\; (2),$$

be the equations of the circular orbit, and let ρ, β, λ, &c. be as before, of which β, λ, L, R are known from observation; therefore, (Art. 273.)

$$x=\rho \cos \beta \cos \lambda + R \cos L,$$
$$y=\rho \cos \beta \sin \lambda + R \sin L,$$
$$z=\rho \sin \beta.$$

Substitute these values in equations (1) and (2), and they become

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* See Lagrange, *Mechanique Analytique*, Vol. II. Gauss *Theoria motus corporum celestium.*

$$\rho\,(\sin\beta+p\cos\beta\sin\lambda+q\cos\beta\cos\lambda)$$
$$+pR\sin L+qR\cos L=0,$$
$$\rho^2+2R\cos\beta\,.\cos(L-\lambda)\,.\,\rho+R^2=a^2,$$

or calling the known coefficients A, B, C, D, E, after dividing by $\sin\beta$,

$$\rho\,.\,(1+Ap+Bq)+Cp+Dq=0\;\ldots\ldots(3),$$
$$\rho^2+2E\rho+R^2=a^2\ldots\ldots(4);$$
$$\therefore\ \rho=\sqrt{a^2+E^2-R^2}-E\ldots\ldots(5),$$

by the solution of (4).

Similarly, if ρ', E' be the values of ρ and E at a second observation,

$$\rho'=\sqrt{a^2+E'^2-R^2}-E'\ldots\ldots(6).$$

But if c be the chord joining the extremities of ρ and ρ'

$$\begin{aligned}c^2&=(x'-x)^2+(y'-y)^2+(z'-z)^2\\&=2a^2-2\,.(xx'+yy'+zz')\\&=2a^2-2\,.\{F\rho\rho'+G\rho R'+K\,.\,\rho'R+LRR'\}\;\ldots\ldots(7),\end{aligned}$$

substituting for x, y, z, x', y', z' their values, and calling the known coefficients F, G, K, L.

The circular sector (s) described round the Sun, whilst the planet moves from the one position to the other

$$=a^2\sin^{-1}.\,\frac{c}{2a},$$

and $s:\pi a^2::$ the observed interval (t) : the planet's period $(a)^{\frac{3}{2}}$, a being expressed in parts of the Earth's radius, and t in parts of a year;

$$\therefore\ a^2.\sin^{-1}\left(\frac{c}{2a}\right)=\pi a^2.\frac{t}{a^{\frac{3}{2}}};$$

$$\therefore t = a^{\frac{3}{2}}.\sin^{-1}\left(\frac{c}{2a}\right) \ldots\ldots\ldots\ldots (8).$$

From equations (5) and (6), with an assumed value of a, let ρ and ρ' be calculated, and thence c by equation (7), then this value of c ought, when substituted in (8), to give the observed time (t), if a is rightly assumed. If a is not rightly assumed, the equation (8) will point out whether the error is in excess or defect, and calling α the correction of a, we have, [differentiating the equations (5) and (6)] $\frac{a}{\rho+E}.\alpha$, and $\frac{a}{\rho'+E}.\alpha$, for the corrections of ρ and ρ', with which we may proceed as before, till a is found to any required degree of accuracy. When a is once found, ρ and ρ' are known from equations (5) and (6), and substituting them in equation (3), and in the equation similarly formed with ρ', two simple equations arise for determining p and q, or the position of the orbit.

278. Since all the planets have small eccentricities, the above mode may be used as an approximate method of determining the mean distance, the place of the node and the inclination of any planetary orbit. When applied to that of Vesta, it gave the inclination within 2′, and the log. of the mean distance within .0056 of its true value.

To find the approximate elements of a comet's orbit.

279. *From three geocentric observations to find the heliocentric and geocentric distances of a comet.*

Let ABC (fig. 54.) be three positions of a comet, t and t' the respective times between the first and second, and second and third observations, t and t' not exceeding 5 or 6 days. Let S be the Sun, and let SB cut AC in D.

$$\text{Then } t : t' :: SAB : SBC :: SAD : SDC, \text{ very nearly}$$
$$:: AD : DC.$$

Similarly, if a, d, c be corresponding positions for the Earth,

$$ad : dc :: t : t' :: AD : DC,$$

and as this proposition must hold when the lines AD, ad are projected on any plane, let them be projected on the ecliptic, and let Dd cut Cc, Aa in O and M.

Let $\rho\rho''\rho'$ be the comet's curtate distˢ. from the $\oplus$ at A, B, C,

$rr''r'$ .... true distˢ. from the $\odot$,

$LL''L'$ the heliocentric longitudes of the Earth,

$\lambda\lambda''\lambda'$ .... geocentric .... Comet,

$bb''b'$ the heliocentric, $\beta\beta''\beta'$ the geocentric latitudes,

and xyz, $x''y''z''$, $x'y'z'$ the co-ordinates of the comet, the Sun being the origin and the line of equinoxes the axis of x.

$$\text{Then } COD = Cce - Dde = \lambda' - \lambda'',$$

$$AMD = Dde - Aae = \lambda'' - \lambda,$$

$$\text{and } CO : CD :: \sin ADO : \sin(\lambda' - \lambda''),$$

$$CD : AD :: t' : t,$$

$$AD : AM :: \sin(\lambda'' - \lambda) : \sin ADO;$$

$$\therefore CO : AM :: t'.\sin(\lambda'' - \lambda) : t.\sin(\lambda' - \lambda'').$$

$$\text{Similarly, } cO : aM :: t'.\sin(\lambda'' - \lambda) : t.\sin(\lambda' - \lambda'');$$

$$\therefore \rho' : \rho :: t'.\sin(\lambda'' - \lambda) : t.\sin(\lambda' - \lambda'');$$

by addition of ratios;

$$\therefore \rho' = \frac{t'.\sin(\lambda'' - \lambda)}{t\sin(\lambda' - \lambda'')}.\rho = m\rho, \text{ suppose.}$$

$$\text{Now } r^2 = x^2 + y^2 + z^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$r'^2 = x'^2 + y'^2 + z'^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$c^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2 \ldots\ldots (3),$$

c being the chord joining the extremities of r and r'.

But $x=\rho\cos\lambda+R\cos L$, $x'=m\rho\cos\lambda'+R'\cos L'$,

$$y=\rho\sin\lambda+R\sin L,\quad y'=m\rho\sin\lambda'+R'\sin L',$$

$$z=\rho\tan\beta,\qquad z'=m\rho\tan\beta'.$$

Therefore substituting these values in (1), (2) and (3), and denoting the known coefficients by A, B, C,

$$r^2=A+B\rho+C\rho^2\,\ldots\ldots\ldots\ldots\ldots\,(4),$$

$$r'^2=A'+B'\rho+C'\rho^2\,\ldots\ldots\ldots\ldots\ldots\,(5),$$

$$c^2=F+G\rho+H\rho^2\,\ldots\ldots\ldots\ldots\ldots\,(6).$$

From these equations with any assumed value of ρ, let r, r', c be calculated, then, if the assumption be correct, the equation

$$t+t'=\frac{365}{12\pi}.\{(r+r'+c)^{\frac{3}{2}}-(r+r'-c)^{\frac{3}{2}}\}\,\ldots\ldots(7),$$

(Art. 269.) must be satisfied.

If this equation is not satisfied, let $\rho+\Delta\rho$ be a corrected value of ρ, then

$$r+\tfrac{1}{2}.\frac{B+2C\rho}{r}.\Delta\rho,\quad r'+\tfrac{1}{2}.\frac{B'+2C\rho}{r'}.\Delta\rho,$$

$$\text{and }\ c+\tfrac{1}{2}.\frac{G+2H\rho}{c}.\Delta\rho,$$

are the corrected values of r, r' and c, which must be proceeded with as before, till values of r, r', c are found, which satisfy equation (7).

280. *From the quantities determined in the preceding proposition, to find the inclination, the place of the node, and the argument of latitude in a comet's orbit.*

Retaining the same construction, let l, l' be the heliocentric longitudes of the comet at the first and third observation, A and A' the geocentric longitudes of the Sun, which are $=180+L$ and $180+L'$; then

$$SN \text{ (fig. 53.)} : EN :: \sin SEN : \sin NSE,$$

$$\text{or } r \cos b : \rho :: \sin (A - \lambda) : \sin NSE = \frac{\rho \,.\, \sin (A - \lambda)}{r \cos \rho}.$$

In this equation A and λ are known from observation, ρ and r have been determined in the preceding article, and b is known from the equation

$$r \sin b = \rho \tan \beta.$$

Hence NSE is known, and adding to this $ME\Upsilon$, the heliocentric longitude of the Earth, the heliocentric *reduced* longitude of the comet (l) is obtained. Let e be the inclination of the comet's orbit, n the longitude of the node, u the argument of latitude at the first observation, and l' and b' the values of l and b at the last observation.

$$\text{Then } \sin (l - n) = \tan b \,.\, \cot e, \quad \text{(Art. 272.)}$$

$$\text{and } \sin (l' - n) = \tan b' . \cot e\,;$$

$$\therefore \frac{\sin (l - n) + \sin (l' - n)}{\sin (l - n) - \sin (l' - n)} = \frac{\tan b + \tan b'}{\tan b - \tan b'},$$

$$\text{or } \frac{\tan \{\frac{1}{2} (l + l') - n\}}{\tan \frac{1}{2} (l - l')} = \frac{\sin (b + b')}{\sin (b - b')};$$

$$\therefore \tan \{\tfrac{1}{2} \overline{l + l'} - n\} = \frac{\sin (b + b')}{\sin (b - b')} \,.\, \tan \tfrac{1}{2} \,.\, (l - l'),$$

from which equation n may be found; and therefore e and u from the equations

$$\cot e = \sin (l - n) \,.\, \cot b,$$

$$\cos u = \cos b \,.\, \cos (l - n).$$

281. *Given two arguments of latitude in a parabolic orbit, and the corresponding distances from the Sun, to find the true anomaly, and the longitude of the perihelion.*

Let $4a$ be the parameter of the orbit, ϖ the distance of the perihelion from the node, and u, u' two arguments of latitude determined as in the preceding article; then $u-\pi$ and $u'-\pi$ are the true anomalies corresponding to the distances r and r', and if

$$u-\pi=v,\quad u'-\pi=v+u'-u;$$

$$\therefore \cos\tfrac{1}{2}.\{v+(u'-u)\}=\sqrt{\frac{a}{r'}},$$

$$\text{and } \cos\tfrac{1}{2}v=\sqrt{\frac{a}{r}};$$

$$\therefore \cos\tfrac{1}{2}(u'-u)-\sin\tfrac{1}{2}(u'-u).\tan\tfrac{1}{2}v=\sqrt{\frac{r}{r'}}, \text{ by division;}$$

$$\therefore \tan\tfrac{1}{2}v=\frac{\sqrt{r'}.\cos\tfrac{1}{2}(u'-u)-\sqrt{r}}{\sin\tfrac{1}{2}(u'-u)}, \text{ which gives } v,$$

and therefore $\pi=u-v$, and the longitude of the perihelion $n+\pi$.

Cor. Since $\tan\frac{1}{2}v$ contains only r, r' and $u'-u$, the true anomaly in a parabolic orbit may be found from two distances, and the angles between.

Since the radius vector in a parabola does not divide the chord accurately as the times, the above mode of finding the elements of a comet's orbit must be incorrect; the error is, however, very small, and the mode of correcting it may be seen in the Essay of Olbers, whence the preceding Articles are extracted.

The elements of a planet's orbit may be obtained much more simply from observations made under favourable circumstances, than by the general methods alluded to in Art. 275, which have been applied to the cases of circular or parabolic orbits; and although in the case of a new planet being discovered, the elements would probably be discovered to a considerable degree of accuracy long before such circumstances would present themselves, yet the following methods are worthy of attention as affording the best means of correcting the nearly known elements.

282. *To find the node of a planet's orbit from observations made on the planet in its node.*

Let S (fig. 55.) be the Sun, E the Earth, and P the planet, when its geocentric latitude, calculated from the observed declination and right ascension is 0, and therefore in its node.

Let $SE = R$, $SP = r$, L the heliocentric longitude of the Earth, l of the node; therefore, $ESP = L - l$, and let E be the elongation of the planet from the Sun = the geocentric longitude of the Sun − that of the planet. Then

$$r : R :: \sin E : \sin SPE :: \sin E : \sin(E+S);$$

$$\therefore\ r.\sin(E+S) = R.\sin E,$$

$$\text{or } r.\sin(E+L-l) = R.\sin E.$$

Let E', L', R', be values of E, L, and R, when the planet returns to the node after one or more revolutions; therefore,

$$r.\sin(E'+L'-l) = R'.\sin E' \ldots\ldots\ldots (2);$$

$$\therefore\ \frac{\sin(E'+L'-l)+\sin(E+L-l)}{\sin(E'+L'-l)-\sin(E+L-l)} = \frac{R'\sin E'+R\sin E}{R'\sin E'-R\sin E};$$

$$\therefore\ \frac{\tan\left(\tfrac{1}{2}.\overline{E'+E+L'+L}-l\right)}{\tan\tfrac{1}{2}.(E'-E+L'-L)} = \frac{R'\sin E'+R\sin E}{R'\sin E'-R\sin E};$$

$$\therefore\ \tan\left(\tfrac{1}{2}\overline{E'+E+L'+L-l}\right)$$

$$= \frac{R'\sin E'+R\sin E}{R'\sin E'-R\sin E}.\tan\tfrac{1}{2}.(E'-E+L'-L),$$

which determines l the longitude of the node.

If no observed right ascension and declination give the planet's geocentric latitude 0, the time when the planet is in its node, which is requisite for finding the values of L and E, must be found by a proportion. Let β and β' be the geocentric latitudes of the planet on successive days before and after passing the node, then, $\beta + \beta' : \beta ::$ time between the two observations : the time between the planet's being in the node and the first observation; which, added to the time of the first observation, gives the time when the planet is in the node; for which time the values of E and L must be found in the same manner by taking the proportional parts.

283. Cor. This method will be slightly incorrect on account of the slow motion of the nodes, which is not noticed; but if the node be determined in this way at different times, and the motion of the node be thence deduced, the motion of the node between the two observations may be found, and if this be called a, and the value $l+a$ be used instead of l in equation (2), the place of the node may be found more accurately.

284. *To find the inclination of a planet's orbit.*

Having found the longitude of the node by the last Article, let E (fig. 56.) be the Earth in the node, S the Sun, P a planet, $P\Pi$ perpendicular to the ecliptic.

Let, as before, $SE\Pi$ the difference of geocentric longitudes of S and $P = E$, $PE\Pi$ the geocentric latitude $= \beta$, and $e =$ the inclination; then, since the angle between the planes $PE\Pi$, $SE\Pi$ is 90^0; therefore, by Napier's Rules,

$$\sin E = \tan \beta . \cot e;$$

$$\therefore \cot e = \sin E . \cot \beta.$$

285. *To find when a planet is in conjunction with the Sun.*

When a planet is in conjunction, its geocentric longitude is the same as that of the Sun.

Let E and E' be the elongations of the planet from the Sun,

T and T' the corresponding times, on successive days before and after conjunction, C the time of conjunction.

Then, considering the change of elongation uniform for the time $T' - T$,

$$E' - E : E :: T' - T : C - T = \frac{E}{E' - E} \cdot (T' - T);$$

$$\therefore C = T + \frac{E}{E' - E} \cdot (T' - T).$$

286. Cor. If β and β' be the geocentric latitudes at the times T and T', and $\beta + \Delta\beta$ at conjunction,

$$\beta' - \beta : \Delta\beta :: T' - T : C - T :: E' - E : E;$$

$$\therefore \Delta\beta = \frac{E}{E' - E} \cdot (\beta' - \beta).$$

Therefore, the geocentric latitude at conjunction

$$= \beta + \frac{E}{E' - E} \cdot (\beta' - \beta).$$

287. Cor. 2. The heliocentric longitude of the planet at conjunction is the same as that of the Earth, and may be found in exactly the same manner as the geocentric latitude. If L and L' be the two longitudes of the Earth on successive days before and after conjunction; the heliocentric longitude of the Earth at conjunction, or the heliocentric longitude of the planet

$$= L + \frac{E}{E' - E} \cdot (L' - L).$$

288. Cor. 3. In exactly the same manner may be found the time when a superior planet is in *opposition to the Sun;*

that is, when it is in the opposite part of the heavens to the Sun, or when the difference between the geocentric longitudes of the planet and Sun is 180^0.

289. Cor. 4. Hence may be obtained the *synodic* time of a planet, which is the interval between two succeeding similar conjunctions. If the two similar conjunctions are taken, separated by a long period, the whole time between the conjunctions divided by the number of revolutions made in the interval, will give the mean synodic time more accurately.

290. *To find a planet's distance from the Sun, and the argument of latitude at the time of conjunction.*

Let r (fig. 57.) be the distance SP of a planet from the Sun at the time of conjunction,

b the heliocentric latitude PSE,

β the geocentric........ $PE\Pi$,

u the argument of latitude NSP.

Let nSN be the line of nodes, and let $SE = R$; then NSE = the heliocentric longitude of the Earth — that of the node = $L - n$.

Therefore, from the solid angle formed at S by the plane angles NSP, NSE, PSE, we have by Napier's rules, since the plane $PS\Pi$ is perpendicular to NSE,

$$\sin (L - n) = \tan b \, . \cot e,$$

$$\text{or } \tan b = \sin (L - n) \, . \tan e, \text{ which determines } b,$$

$$\text{and } \cot u = \sin e \, . \cot (L - n) \ldots\ldots\ldots\ldots\ldots u,$$

$$\therefore (R - r \, . \cos b) \, . \tan \beta = P\Pi = r \, . \sin b;$$

$$\therefore r = \frac{R \, . \tan \beta}{\sin b + \cos b \, . \tan \beta}.$$

291. *Given three distances of a planet from the Sun, and the corresponding arguments of latitude; to find the place of the perihelion and the true anomaly at the first observation.*

Let u, $u+\alpha$, $u+\beta$, be the three arguments of latitude, r, r', r'', the corresponding distances from the Sun, determined by observations made at three different conjunctions, as in the last Article, then, if v be the true anomaly corresponding to the distance r, $v=u-\pi$, π being the distance of the perihelion from the node. The true anomalies at the second and third observations are $v+\alpha$, $v+\beta$; therefore,

$$r.(1+e\cos v)=a.(1-e^2)\ldots\ldots(1),$$

$$r'.\{1+e\cos(v+\alpha)\}=a.(1-e^2)\ldots\ldots(2),$$

$$r''.\{1+e\cos(v+\beta)\}=a.(1-e^2)\ldots\ldots(3);$$

$$\therefore\ r'-r+e.\{r'\cos(v+\alpha)-r\cos v\}=0,\text{ from (1) and (2)},$$

$$r''-r+e.\{r''\cos(v+\beta)-r\cos v\}=0,\text{ from (1) and (3)}.$$

Therefore, eliminating e between these equations,

$$(r'-r).\{r''\cos(v+\beta)-r\cos v\}=(r''-r)\{r'\cos(v+\alpha)-r\cos v\};$$

therefore, expanding $\cos(v+\alpha)$ and $\cos(v+\beta)$, and dividing by $\cos v$,

$$(r'-r).\{r''\cos\beta-r''\sin\beta.\tan v-r\}$$

$$=(r''-r).\{r'\cos\alpha-r'\sin\alpha\tan v-r\};$$

$$\therefore\ \tan v=\frac{rr'-r''r+(r''-r).r'.\cos\alpha-(r'-r)r''\cos\beta}{(r''-r).r'\sin\alpha-(r'-r).r''\sin\beta},$$

which determines v.

$$\text{Therefore, } e=\frac{r-r'}{r'\cos(v+\alpha)-r\cos v},$$

$$a=\frac{r}{1-e^2}.(1+e\cos v),$$

and the longitude of the perihelion ($u - v + n$) are determined.

292. The values of v, e, and a, contain only the three distances and the angles α and β, and may, therefore, be found from these data; but the argument of latitude u is requisite for determining the place of the perihelion.

293. *To find a planet's period.*

Let the time when the planet is in its node be observed, then, if the same observations are made after one or more revolutions of the planet, the whole time divided by the number of revolutions is the period with respect to the node; this is not the sidereal period, or the time from a fixed star till it returns to it again, on account of the regression of the node between the observations. If this regression, calculated as in Art. 283, be a, and n be the number of revolutions, $n \,.\, 360^0 - a$ is the whole angle described by the planet;

$\therefore\ n \,.\, 360^0 - a : 360 ::$ the observed time : the sidereal periodic time.

294. The period may be also found from the synodic time; for if this time be called S, and P and p be the sidereal periodic times of the Earth and planet, all expressed in days $\frac{360^0}{p}$ and $\frac{360^0}{P}$ are the mean daily motions; therefore $\frac{360^0}{p} - \frac{360^0}{P}$ is the mean daily separation of the Earth and planet, which in the time S ought to be 360^0;

$$\therefore \left(\frac{360^0}{p} - \frac{360^0}{P}\right) S = 360^0;$$

$$\therefore \frac{1}{p} - \frac{1}{P} = \frac{1}{S};$$

$$\therefore p = \frac{PS}{P+S}.$$

p is here supposed less than P, or the planet an inferior one; if the planet be superior, it may be proved in the same way that

$$p = \frac{PS}{S-P}.$$

295. The period might have been obtained from the mean distance a, found in the preceding Article, by the proportion P : 1 year :: $a^{\frac{3}{2}}$: (rad. of Earth's orbit)$^{\frac{3}{2}}$; but, on account of the errors to which the determination of a is liable, P can be found more accurately by the method here given, and then the above proportion will serve to determine a.

296. As none of the preceding methods are exact, the elements of the planets, so determined, will be subject to slight errors, and will require subsequent corrections, which must be applied by comparing the observed places with the places calculated from the nearly known elements, and thus determining the errors in geocentric latitude and longitude. If the equations connecting the geocentric and heliocentric places of a planet are differentiated, the errors in the heliocentric place, corresponding to the errors in the geocentric place, are obtained, and again, by differentiating the equations by which the heliocentric place depends on the elements, the errors in the heliocentric place are obtained in terms of the errors in the elements, and thus an equation is obtained connecting the observed errors in the geocentric place with the errors of the elements, and from as many equations of this kind as there are errors, the errors may be found. This is a brief outline of the method which is attended with considerable difficulty in the details. See Littrow, Vol. II. p. 201.

297. *To find the geocentric motion of a planet, supposing the Earth and planet to describe circles in the ecliptic.*

Let a and ρ be the distance of a planet from the Sun and Earth, the Earth's distance from the Sun being (1). Then, l and L being as before the heliocentric longitudes of the planet and Earth, and λ the geocentric longitude of the planet,

$$a \cos l - \cos L = \rho \cos \lambda, \text{ Art. 269.}$$

$$a \sin l - \sin L = \rho \sin \lambda;$$

$$\therefore \tan \lambda = \frac{a \sin l - \sin L}{a \cos l - \cos L};$$

$$\therefore \frac{d\lambda}{\cos^2 \lambda}$$

$$= \frac{(a \cos l - \cos L).(a \cos l.dl - \cos L.dL) + (a \sin l - \sin L)(a \sin l.dl - \sin L.dL)}{(a \cos l - \cos L)^2}$$

$$= \frac{\{a^2 - a \cos (L-l)\}\, dl + \{1 - a . \cos (L-l)\}\, dL}{(a \cos l - \cos L)^2}.$$

$$\text{But } dL : dl :: a^{\frac{3}{2}} : 1;\ \therefore dL = a^{\frac{3}{2}}.dl;$$

therefore, substituting this value, and making

$$\frac{\cos \lambda}{a \cos l - \cos L} = P,$$

$$d\lambda = P^2 . \{a^2 + a^{\frac{3}{2}} - (a + a^{\frac{5}{2}}) . \cos \overline{L - l}\}\, dl.$$

298. Cor. 1. When an inferior planet is in the conjunction nearest the Earth, or a superior planet in opposition, $L - l = 0$,

$$\therefore d\lambda = P^2 a . (a + a^{\frac{1}{2}} - 1 - a^{\frac{3}{2}}) . dl$$

$$= P^2 a (a - 1) . (1 - a^{\frac{1}{2}}) . dl,$$

which is negative, whether a be greater or less than 1, or the planet's motion is retrograde.

When a superior planet is in conjunction, or an inferior planet in the conjunction farthest from the Earth, $L - l = 180^0$, and $\cos (L - l) = -1$; therefore $d\lambda$ is positive, or the motion is direct.

299. Cor. 2. When $d\lambda = 0$, the planet appears stationary. In this case

$$\cos (L - l) = \frac{a + a^{\frac{1}{2}}}{1 + a^{\frac{3}{2}}}.$$

If t be the time from syzygy when this happens, and m and m' be the daily motions of the planet and the Earth in longitude,

$$L - l = (m - m') . t;$$

$$\therefore t = \frac{L - l}{m - m'}.$$

Therefore a planet's motion appears retrograde whilst it describes an arc

$$= 2m . \frac{L - l}{m - m'},$$

and progressive whilst it describes the arc,

$$360^\circ - 2m . \frac{L - l}{m - m'}.$$

300. Cor. 3. If the Earth were at rest, or $dL = 0$, we should have

$$d\lambda = P^2 . \{a^2 - a \cos (L - l)\},$$

which could never equal 0 for a superior planet; therefore, as the superior planets often appear stationary, this affords another presumption that the Earth moves.

301. *From the observed elongation of a planet when stationary, to find its distance from the Sun upon the hypothesis of the preceding article.*

Let E (fig. 55.) be the Earth, S the Sun, P the planet when stationary, and let $SP = a$, $SE = 1$; then

$$a : 1 :: \sin E : \sin (E + S) :: 1 : \cos S + \sin S . \cot E;$$

$$\therefore \tan E = \frac{a \sin S}{1 - a \cos S}.$$

$$\text{But } \cos S = \cos(L - l) = \frac{a + a^{\frac{1}{2}}}{1 + a^{\frac{3}{2}}};$$

$$\therefore \tan E = a\frac{\sqrt{1 - \left(\frac{a + a^{\frac{1}{2}}}{1 + a^{\frac{3}{2}}}\right)^2}}{1 - a\,.\,\frac{a + a^{\frac{1}{2}}}{1 + a^{\frac{3}{2}}}} = a\frac{\sqrt{1 - a^2 + a^3 - a}}{1 - a^2}$$

$$= a\,.\,\frac{\overline{1 - a}\sqrt{1 + a}}{1 - a^2} = \frac{a}{\sqrt{1 + a}};$$

$$\therefore a^2 = \overline{1 + a}\,.\,\tan^2 E;$$

$$\therefore a^2 - a \tan^2 E + \tfrac{1}{4}\,.\,\tan^4 E = \tan^2 E + \tfrac{1}{4}\,.\,\tan^4 E;$$

$$\therefore a = \tfrac{1}{2}\,.\,\tan^2 E + \tfrac{1}{2}\tan E\,.\,\sqrt{4 + \tan^2 E}.$$

By this means the mean distance of Pallas was approximated to as that planet was stationary soon after it was first discovered.

302. The planets are too distant to present to the naked eye any very obvious change of appearance in their different positions with regard to the Sun; but Venus, Mercury and Mars, which are situated most favourably for such observations, shew on a small scale the same phases as the Moon does, or such as would be shewn by opaque spherical bodies illuminated solely by the rays of the Sun.

303. *To find the phases of an opaque heavenly body illuminated solely by the Sun's rays.*

Let S (fig. 58.) be the Sun, E the Earth, P the center of a planet, *adcb* a section of it made by the plane SPE; then the plane *amcn* drawn perpendicular to EP, cuts off the visible hemisphere of the planet, and the plane *mdnb* perpendicular to SP cuts off the illuminated hemisphere; therefore *mcnbm* is the visible illuminated part of the planet, which will be seen from E projected on the plane $amcn$; therefore the boundary mbn will become an ellipse, whose

$$\text{major axis : the minor} :: 1 : \cos bPc :: 1 : \sin cPS$$
$$:: 1 : \cos SPe.$$

If therefore 2Δ be the visible disk of the planet, or the circle $amcn$, the area of the semi-ellipse, which is the projection of mbn, will equal $\Delta . \cos SPe$. Therefore the illuminated part of the planet

$$= \Delta - \Delta . \cos SPe = \Delta . \text{ver sin } SPe,$$
$$\propto \text{ver sin of the exterior angle of elongation.}$$

If the planet be supposed an inferior planet, and its orbit to coincide with the ecliptic, the exterior angle of elongation is 0 at the nearer conjunction, and increases to 180^0 at the superior conjunction; therefore the illuminated part of the disk is 0 at conjunction, and increases gradually to the farther conjunction, where it equals 2Δ, or the planet's disk. As the angle of elongation becomes greater than 180^0, the illuminated part diminishes, till the angle of elongation $= 360^0$, or the planet is again in conjunction, when no part of the disk is illuminated.

The planet's orbit is here supposed to coincide with the ecliptic, but as that is not exactly the case for either of the inferior planets Venus and Mercury, the angle of elongation never vanishes, unless the planet be in its node at the time of conjunction; therefore, except in this case, some portion of the disk is always illuminated.

In the same manner may the phases of a superior planet be found from the above expression for the illuminated part.

304. On account of its proximity to the Earth, Venus is generally the most conspicuous of the planets, appearing like a bright star in the heavens. When it is near the Sun, and to the east of it as the Earth revolves from west to east the horizon of any place revolving in the same manner will pass through Venus

soon after passing through the Sun, or Venus will set soon after the Sun, in which case it is called an Evening Star. When Venus is a little to the west of the Sun, from the same cause the horizon will pass through Venus, a little time before it passes through the Sun, or Venus will rise a little before the Sun, and is then called a Morning Star.

305. *To find when Venus appears brightest, upon the hypothesis of Venus, and the Earth describing circular orbits about the Sun.*

Let S (fig. 59.) be the Sun, P Venus, E the Earth.

Then the illuminated part of the disk varies as

$$1 - \cos pPE \propto 1 + \cos SPE,$$

and as the intensity of light from any object varies inversely as the square of the distance; therefore

$$\text{the brightness of Venus} \propto \frac{1 + \cos SPE}{PE^2}.$$

Let $PE = x$, $PS = a$, $SE = 1$; then, the brightness

$$\propto \frac{1}{x^2}\cdot\left(1 + \frac{a^2 + x^2 - 1}{2ax}\right) \propto \frac{1}{x^2} + \frac{a^2 - 1}{2ax^3} + \frac{1}{2ax};$$

Therefore when the brightness is a maximum,

$$\frac{2}{x^3} + 3\,.\frac{a^2 - 1}{2ax^4} + \frac{1}{2ax^2} = 0;$$

$$\therefore x^2 + 4ax + 3\overline{a^2 - 1} = 0;$$

$$\therefore x = \sqrt{a^2 + 3} - 2a,$$

the negative value of the root being inadmissible.

In the following table, the elements of the new planets, Vesta, Juno, Ceres and Pallas, which are not known with great accuracy, are reduced to the epoch of midnight, December 31, 1819, Paris time. All the other quantities in the table are for the epoch of December 31, 1800.

TABLE OF THE SOLAR SYSTEM.

Planet.	Sidereal Period.	Mean Distance.	Eccentricity.
Mercury . .	87^{d}.9692580	.3870981	.20551494
Venus . . .	224 .7007869	.7233316	.00686074
Earth	365 .2563835	1.0000000	.01685318
Mars	689 .9796458	1.5236923	.0933070
Vesta	1325 .7431	2.36787	.089130
Juno	1592 .6608	2.669009	.257848
Ceres	1681 .3931	2.767245	.078439
Pallas . . .	1686 .5388	2.772886	.241648
Jupiter . . .	4332 .5841212	5.202776	.0481621
Saturn . . .	10759 .2198174	9.5387861	.0561505
Uranus . . .	30686 .8208296	19.182390	.0466108

	Mean Longitude.	Long. Perihelion.	Long. Node.
Mercury . .	163° 56′ 27″	74° 21′ 47″	45° 57′ 31″
Venus	10 44 35	128 37 1	74 52 40
Earth	100 9 13	99 30 5	. . . .
Mars	64 7 2	332 24 24	48 1 28
Vesta	278 21 57	249 33 24	103 11 30
Juno	200 9 32	53 33 58	171 9 58
Ceres	123 39 41	147 7 31	78 53 24
Pallas . . .	108 18 27	121 7 4	172 39 27
Jupiter . . .	112 12 36	11 8 35	98 25 34
Saturn . . .	135 20 22	89 8 58	111 35 47
Uranus . . .	177 47 18	167 21 42	72 51 14

TABLE OF THE SOLAR SYSTEM.

Planet.	Inclination.	Secular Variations of		
		Long. of Perihel.	Long of Node.	Inclination.
Mercury	7° 0′ 0″	+643″.56	− 782″.27	+18″.183
Venus . .	3 23 25	−267 . 60	−1869 . 80	−4 . 552
Earth . .	. . . . .	+1177 . 81	. . . .	. . . .
Mars . .	1 51 0	+1582 . 43	−2328 . 44	−0 . 152
Vesta . .	7 8 9	unknown.	. . . .	. . . .
Juno . . .	13 4 45	. . . .	. . . .	. . . .
Ceres . .	10 37 26	12130	148	440
Pallas . .	34 34 55	unknown.	. . . .	. . . .
Jupiter .	1 18 52	+663 . 86	−1577 . 57	−22 . 609
Saturn . .	2 29 38	+1943 . 07	−2266 . 47	−15 . 512
Uranus .	0 46 25	+238 . 62	−3597 . 96	+ 3 . 153

	Time of Rotation.	Masses.	Volume.	True Diameters.
Sun . . .	23^d 10^h	329630. 000	1328460.1	109.93
Mercury	24^h . .	0.1627	.1	.39
Venus . .	23 30′	.9243	.9	.97
Earth . .	23 56	1.	1.	1.
Mars . .	24 40	.1294	.2	.56
Jupiter .	9 52	308. 940	1470.2	11.56
Saturn . .	10 16	93. 271	887.3	9.61
Uranus .	unknown.	1. 690	77.5	4.26

CHAP. XV.

THE MOON AND SATELLITES.

307. To a spectator on the Earth's surface the Moon appears to move in the same direction as the Sun, or according to the order of the signs, and by this motion to complete its circuit round the heavens in about twenty-seven days. Its orbit lies within that which the Sun appears to describe round the Earth; because it often passes between the Sun and Earth, intercepting all or a portion only of the Sun's light, and causing eclipses of that luminary, and its elongation from the Sun passes through all degrees of magnitude from 0^0 to 180^0. The Sun's motion is spoken of here and in other places as explaining appearances, but as the Earth really moves round the Sun, the Moon must be supposed, in order to produce the same appearances, to revolve round the Earth in a month, whilst the Earth itself is carried over an arc of its annual orbit round the Sun.

From the comparison of the Moon's parallax, which is about 1^0, with that of the Sun ($8''.47$), we find that the distance of the Moon from the Earth : that of the Sun :: $8''.47 : 1^0 :: 1 : 450$. This makes the Moon's distance $=60$ radii of the Earth, nearly.

308. The moon, like the planets, is a round opaque body, deriving all its light from the Sun; for the various appearances which the Moon affords are exactly such as would be presented by an opaque body illuminated by the Sun's rays in the different parts of its orbit.

309. The Moon's latitude changes continually during the

course of a month. It is sometimes 0, when the Moon must be in the ecliptic; after this it gradually increases, till, in about 7 days it attains its maximum $5^0\ 8'\ 47''$, when it again diminishes by the same law by which it increased, and at the end of 14 days crosses the ecliptic again, and moves in the same manner on the south side of the ecliptic. By the comparison of the observed latitudes of the Moon it may be proved, as it was for the Sun (Art. 70.), that the Moon's orbit lies in a plane cutting the ecliptic in the points where the Moon's latitude is 0.

310. *To find the inclination of the Moon's orbit to the ecliptic, and the place of the line of nodes.*

Let β, and β' be two latitudes of the Moon on successive days, before and after passing the node. l and l' the corresponding longitudes on the ecliptic, and n the longitude of the node; then, considering the change of latitude and longitude proportional to the time,

$$\beta + \beta' : \beta :: l' - l : n - l = \frac{\beta}{\beta + \beta'} . (l' - l);$$

$$\therefore\ n = l + \frac{\beta}{\beta + \beta'} . (l' - l).$$

If ν be the inclination, as in Art. 290,

$$\sin (l - n) = \tan \beta . \cot \nu;$$

$$\therefore\ \cot \nu = \sin \overline{l - n} . \cot \beta.$$

311. The inclination may also be deduced by observing the Moon's latitude on successive days when near its maximum, and determining by interpolations the maximum latitude, which will be the inclination of the Moon's orbit.

312. If the Moon's nodes are determined at different times, they are found to be not fixed; but to have an uniform retrograde motion of $19^0\ 20'\ 33''.46$ in 365 days, with respect to the stars, and, therefore, a tropical motion of $19^0\ 19'\ 43''.36$ in the

same time, and besides this mean motion are subject to inequalities, of which the greatest is $1^0\ 30'\ 26''.\sin 2.(\odot - \Omega)$; so that if Ω be the longitude of the ascending node, determined from its position at a known epoch, and the uniform increase of $19^\circ\ 10'\ 43''$ in 365 days, the true longitude of the node

$$= \Omega + 1^0\ 30'\ 26''.\sin 2(\odot - \Omega).$$

The inclination has also a small inequality

$$= 8'\ 47''.\cos 2(\odot - \Omega),$$

to be added to the mean inclination which remains constant, if the secular change in the position of the ecliptic be omitted.

313. *To find the Moon's synodic and sidereal periods.*

The time of the Moon's conjunction with the Sun may be found as that of Venus was (Art. 285.), and from the observed interval between two conjunctions, the synodic period may be deduced. This will be variable on account of the inequalities of the Sun and Moon's motions; but if the whole time between two distant conjunctions be divided by the number of revolutions made by the Moon in the interval, the mean synodic time will be determined with great accuracy.

The synodic time thus deduced

$$= 29^d\ 12^h\ 44'\ 2''.82,$$

and substituting this quantity for S in the equation $p = \frac{P.S}{S+P}$, where P is a sidereal year, we obtain the mean sidereal period

$$p = 27^d\ 7^h.43.$$

If P be a tropical year, the tropical period of the Moon will be found.

314. As there exist very ancient records of solar eclipses, which can only take place at the time when the Moon is near

conjunction, the mean synodic times, and mean motion of the Moon, may be deduced from the comparison of some of these, and it is found in every case, that the more ancient the times of conjunction from which the synodic period is deduced, the greater is the period, so that the Moon's mean motion must have been continually accelerated. Laplace, who first detected the cause and law of this acceleration from the theory of gravity, has proved that the quantity

$$10''.207\,t^2 + 0''.00185\,t^3,$$

must be added to the Moon's mean longitude, to give the true mean longitude for the year $1800 + t$.

315. *To find the elements of the lunar orbit.*

The Moon may be proved in the same manner as the Sun, by observations of the apparent diameter to describe an ellipse, and as the variation of this diameter is much greater than the Sun's, this circumstance may be made use of in ascertaining the elements with some accuracy. The *Place of the Node*, the *Inclination of the Orbit*, and the *Mean Motion*, have already been determined. To find the *Longitude of the Apogee*, let the Moon be observed when its diameter is least, then the longitude of the Moon is the longitude of the apogee, and if the place of the apogee be determined in this manner after several revolutions of the Moon, its motion may be determined, and the place found with more accuracy.

The eccentricity $= \dfrac{\Delta - \Delta'}{\Delta + \Delta'}$, Δ and Δ' being the Moon's greatest and least apparent diameters.

The mean distance may be found from the parallax. If p be the horizontal parallax, when the Moon's apparent diameter is δ, determined as Art. 173.

$$\text{the parallax at the least distance } = p \cdot \frac{\Delta}{\delta},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{ greatest} \ldots\ldots = p \cdot \frac{\Delta'}{\delta}.$$

$$\text{Therefore the least distance} = \text{rad of } \oplus \,.\, \frac{\delta}{p\Delta},$$

$$\text{............ greatest......} = \text{rad of } \oplus \,.\, \frac{\delta}{p\,.\,\Delta};$$

$$\text{therefore the mean distance} = \tfrac{1}{2}\,\text{rad}\,.\,\oplus\,.\,\frac{\delta}{p}\,.\left\{\frac{1}{\Delta}+\frac{1}{\Delta'}\right\}.$$

The longitude of the Moon at a given epoch is known from the observed time of the Moon's being in its node.

The numerical values of the lunar elements at the beginning of the nineteenth century, are, according to Laplace, at midnight, Dec. 31, 1800, Paris time.

The mean longitude of the Moon ...	111^{0}.613302
.... perigee ..	266 .10795
.... ascend. ☊	13 .9183
The mean inclination	5 .14395
The eccentricity	0 .548553
The greatest apparent diameter	0 .55863
The least	0 .48942
The mean horizontal parallax	0 .95949
The sidereal revolution of the perigee	3232^{d}.575614
............................node	6793 .39081
The mean distance of the Moon	60.2379 . rad ⊕.

316. Cor. The motion of the apogee in longitude found by determining its place at different times, is 40° 39′ 45″.79 in 365 days. To the longitude determined from its place at a known epoch, and to this mean motion must be added 22′ 17″ sin Sun's mean anomaly, in order to obtain the true longitude of the apogee.

317. The motion of the apogee and node, besides their mean values and periodical equations, have also secular equations which are to the secular acceleration of the Moon's mean motion (Art. 314.) as 3 : 1 and .74 : 1, respectively.

By the longitude of the Moon is generally understood the longitude in its orbit, from which the reduced longitude on the ecliptic may be easily obtained.

318. *The equations of the Moon's motion.*

If the Moon and Earth were the only bodies of the system, the Moon would describe an accurate ellipse about the Earth, and the apogee and nodes would be stationary; but in consequence of the attractions of the other bodies of our system, and of the Sun in particular, this motion is much disturbed, and the longitude of the Moon, determined from the elements upon the hypothesis of an elliptic motion, must receive between 30 and 40 corrections, in order to determine the true place of the Moon, with the necessary degree of accuracy. The form of these corrections or *equations* as they are usually called, is determined by Physical Astronomy, and the coefficients are calculated from observation. The three greatest equations, and the first discovered, are

$$\text{The evection} = 1^{0}\ 20'\ 30''\ \sin 2\{(☽ - \odot) - M\},$$

$$\text{the variation} = 35'\ 42''\ \sin 2(☽ - \odot),$$

$$\text{the annual equation} = 11'\ 12''\ \sin m,$$

where ☽ is the Moon's mean longitude, ⊙ the Sun's true longitude, and m and M the mean anomalies of the Sun and Moon.

To use these equations, let ☽ be the Moon's mean longitude calculated from the mean motion at any time.

Then the mean longitude corrected for the evection and variation

$$☽' = ☽ - 1^{0}\,20'\,30''\sin\{2(☽ - \odot) - M\} - 35'\,42''\sin 2(☽ - \odot).$$

Let E be the equation of the center, calculated as in Art. 240. from this mean longitude, and the place of the perigee; then the true longitude

$$☽'' = ☽' - E.$$

And the true longitude corrected for the annual equation

$$☽''' = ☽'' - 11' \ 12'' \sin m.$$

319. *The time of the Moon's rotation equal the time of its revolution round the Earth.*

The Moon seen through a telescope, appears covered with a great number of spots resembling excavations or prominences, they remain fixed *nearly* in the same position, and have been laid down in maps. These maps serve for all times, and therefore prove that the same face of the Moon is always turned towards the Earth, from whence it follows, that the Moon turns round its own axis from west to east in the time of its motion round the Earth. For an observer placed in the Moon's center as the Moon revolves round the Earth, would attribute all the motion observed to the Earth, and the line joining the centers of the Earth and Moon would appear to him to revolve round the heavens from west to east in a month. But as the same face of the Moon is always turned towards the Earth, this line must always cut the Moon's surface in the same point, which would be impossible, unless the Moon's surface turned round an axis perpendicular to the plane of the Earth's apparent motion in the course of a month.

The greater part of the Moon's face turned towards the Earth remains the same, but parts near the edge are brought into view, and carried out periodically, and these effects are called *Librations*.

320. *The librations of the Moon.*

The tendency of the Moon's motion in its orbit, is to bring the western limb into view, and to carry out the eastern limb, which is contrary to the effect produced by rotation; and in the course of a whole revolution, as much is carried out of sight by the rotation, as is brought in by the revolution. But as the motion of rotation is probably constant, whilst the angular motion round the Earth varies; therefore when the Moon's angular velocity is less than the mean, more of the eastern limb ought to be brought into view by the rotation, than is carried out by the Moon's motion in its orbit, and less when the Moon's motion is greater than the mean. This effect coincides with observation, and is called the Moon's *Libration in Longitude*.

The *Libration in Latitude* is an oscillatory motion of the parts of the Moon's disk near its north and south poles, and is caused by its axis of rotation, not being exactly perpendicular to the plane of the Moon's orbit. In consequence of this, when the north pole is turned towards the Earth, the parts adjacent to it are visible; and at the end of 14 days, when the north pole is turned from us, these parts are withdrawn, and the parts near the Moon's south pole are brought into view.

A third libration called *Diurnal Libration*, is due to parallax, in consequence of which, when the Moon rises, parts of the upper limb are seen, which would not be seen from the Earth's center, and which disappear as the Moon ascends.

Besides these librations which are only apparent, and do not affect the Moon's uniform rotation, there is also a real motion of the Moon's equator. The inclination of the Moon's equator to the ecliptic, may be determined in the same manner as the inclination of the Sun's equator, Art. 340, and it has been found, that a plane drawn through the Moon's center, parallel to the ecliptic, lies between the plane of the Moon's equator and the plane of the Moon's orbit, and that these three planes always intersect in the same line; therefore as the line of nodes of the Moon's orbit makes a complete revolution in 6793^{d}, the intersection of the Moon's equator and the ecliptic must revolve in the same time, and the poles of the Moon's orbit and equator must describe small circles round the pole of the ecliptic in such a manner, that these three poles lie always in the same great circle.

321. *The Moon's phases.*

Let S, M, E (fig. 60.) be the respective centers of the Sun, Moon and Earth, and Mm perpendicular to SE; then, as in Art. 303. if $2\,\Delta$ be the Moon's disk, and P the phase, or illuminated part of it,

$$P = \Delta \,.\, \text{ver-sin}\, SMO = \Delta \,.\, \text{ver-sin}\,(S+E).$$

But $S = \dfrac{Mm}{SM}$ nearly, $= \dfrac{EM \,.\, \sin E}{SE}$, $= 60 \,.\, \dfrac{\text{rad}\,\oplus}{SE} \,.\, \sin E$,

$$\text{and } \frac{\text{rad}\,\oplus}{SE} = \text{the Sun's parallax } 8''.47;$$

$$\therefore S = 8'.47 \,.\, \sin E,$$

$$\text{and therefore } P = \Delta \,.\, \text{ver-sin}\,(E + 8'.47 \,.\, \sin E),$$

where E is determined by the equation

$$\cos E = \cos\beta \,.\, \cos(\text{☽} - \odot),$$

☽ being the Moon's *reduced* geocentric longitude, and β the geocentric latitude.

In conjunction,

$$\text{☽} - \odot = 0, \text{ and } E = \beta;$$

$$\therefore P = \Delta \,.\, \text{ver-sin}\,(\beta + 8'.47 \,.\, \sin\beta),$$

which is not 0, unless the Moon be in its node.

As E increases, the illuminated part increases, and when

$$E + 8'.47 \,.\, \sin E = 90^0,$$

half the disk is illuminated. E is, in this case, evidently less than 90^0.

$$\text{When } \text{☽} - \odot = 90^0,\ E = 90,$$

and P is greater than Δ, and goes on increasing till opposition, when

$$P = \Delta \,.\, \text{ver-sin}\,(180^0 - \beta - 8'.47 \sin\beta),$$

which equals the Moon's disk, if the Moon be then in its node.

322. Cor. When half the Moon's disk is illuminated,

$$SME = 90^0,$$

and the boundary of the illuminated part is a right line; if, therefore, the angle E be observed when the boundary of light is a straight line,

$$SE : EM :: 1 : \cos E,$$

or the ☽'s parallax : the ☉'s :: 1 : cos E.

This method of determining the Sun's parallax from the Moon's, was employed by the ancients, but does not admit of great accuracy, from the difficulty of observing the exact time when the boundary of light and darkness is a right line.

323. The Earth, to an inhabitant of the Moon, must present nearly the same phases as the Moon does to us, and if P' be the visible illumined part of the Earth's disk, and $2\Delta'$ the apparent disk seen from the Moon,

$$P' = \Delta' . \text{ver sin}\,(180 - E) = \Delta' . (1 + \cos E),$$

$$\text{and } P = \Delta . \text{ver sin}\,(S + E) = \Delta . (1 - \cos E), \text{ nearly,}$$

since the greatest value of S is $8'.47$.

Hence, when

$$E = 0, \; P = 0, \text{ and } P' = 2\Delta'.$$

As E increases, P increases, and P' diminishes, and when

$$E = 180^0, \; P = 2\Delta, \text{ and } P' = 0.$$

The pale light observed on the Moon in conjunction, is caused by the reflected light from the Earth, then at the full, falling on the dark side of the Moon.

323. *To find the retardation of the Moon's rising on successive nights.*

Let HO (fig. 61.) be the horizon, EQ the equator, Z the zenith, AB the Moon's orbit, A the place of the Moon at rising,

B the place in its orbit where it rises the next day, then if Bb be a parallel of the equator, b will be the place of the horizon on which it rises, and Dd will represent the difference of the times of rising on succeeding days, pBD, pbd being circles of declination through B and b.

Let AB the Moon's daily motion $= m$,

BAO the inclination of the Moon's orbit to the horizon $= n$,

and $ZE = 90^0 - HE = pH$, the latitude of the observer, $= l$.

Then considering the small triangle BAb as rectilinear,

$$Bb : m :: \sin n : \sin BbA \text{ or } \cos Hbp,$$

$$\text{and } Dd : Bb :: 1 : \cos \delta;$$

$$\therefore Dd : m :: \sin n : \cos \delta . \cos Hbp.$$

But in the right-angled triangle Hpb,

$$\sin l = \sin pb . \sin Hbp = \cos \delta . \sin Hbp;$$

$$\therefore \sin Hbp = \frac{\sin l}{\cos \delta},$$

$$\text{and } \cos Hbp = \frac{\sqrt{\cos^2 \delta - \sin^2 l}}{\cos \delta};$$

$$\therefore Dd = \frac{m . \sin n}{\sqrt{\cos^2 \delta - \sin^2 l}}.$$

324. Cor. Hence may be explained the phenomenon of the *Harvest Moon.*

If the Moon's orbit be supposed to coincide with the ecliptic, which it does nearly, $\sin n$ is least, when the Moon rises in Aries (Art. 84.); therefore the numerator of the above expression is then least, and because $\cos^2 \delta = 1$, the denominator is then greatest; therefore on both accounts Dd is least, and if the Sun be at the same time in Libra, the Moon is then at the full; therefore the full Moon, which takes place near the autumnal equinox, rises nearly at the same time for several nights, and as this is near the time of harvest in north latitudes, it is called the Harvest Moon.

The inclination of the Moon's orbit to the horizon continually increases from the rising of its ascending node till the setting, when it is greatest; therefore, the difference of the times of setting of the full Moons, when the Sun is in Libra, is the greatest.

325. For the same reason that the solar days are longer in summer than in winter, the Moon will be longer above the horizon when on the north side of the equator than when on the south; therefore in winter, the full Moons which are opposite to the Sun, being on the north side of the equator, will be longer above the horizon than in summer, when their light is less needed.

326. *To find the height of a lunar mountain.*

Let M (fig. 62.) be the summit of a mountain illuminated by a ray of light from the Sun S, which touches the Moon's disk in L; then, to a spectator at the Earth E, LM will appear as a dark line, projected on the plane of vision by the perpendicular lines EMn, Lo, and may be measured by a micrometer.

Let C be the Moon's center, join CM, and measure SEM the angle of elongation of S from M; then, since the distance of the Sun from the Earth and Moon is very great compared with their distance from each other, ES may be considered parallel to LM, and the angle $LMn = SEM = E$.

Let $no (= Lr)$, the projection $= a$, a being measured by a micrometer in parts of the Moon's disk, $r =$ the radius of the Moon, and $x =$ the height of the mountain; then, because LM is a tangent at L,

$$(2r + x) . x = LM^2 = \frac{a^2}{\sin^2 E};$$

$$\therefore x = \frac{a^2}{2r . \sin^2 E}, \text{ neglecting } x^2.$$

Cor. If $E = 90^0$, $x = \frac{1}{2} \cdot \frac{a^2}{r}$.

Some of the mountains thus measured in the Moon are found to be near five miles high, which is much greater compared with the Moon's diameter, than the height of any terrestrial mountain is, compared with the Earth's.

Jupiter's Satellites.

327. Jupiter, when seen through a telescope, is seen accompanied by four small bodies which never separate far from it, and appear to revolve round it from west to east, becoming invisible when they enter the west side of Jupiter's shadow, and again emerging on the east. From this it is evident that these bodies which are called satellites are opaque bodies, serving the purpose of Moons to Jupiter, and revolving round it whilst Jupiter moves in his orbit, in the same manner as the Moon does round the Earth. The beginning and end of the eclipses of the third and fourth satellite can generally be observed from the Earth, but the two other satellites are so near to the body of Jupiter that the immersion only can be seen when the Earth is west of the line joining Jupiter and the Sun, and the emersion when it is east, the satellites being hidden by Jupiter's body, or suffering an occultation, at the time of emersion in the first case, or of immersion in the second.

328. *To find the period of a satellite round Jupiter.*

Let the beginning and end of an eclipse of the satellite be observed, then, the middle of the eclipse may be found, which is nearly the time of the planet's opposition. If the time of another opposition be found in the same manner, the whole time between, divided by the number of oppositions in the interval, gives the synodic time of the satellite (S). From this, the sidereal period round Jupiter $= \frac{PS}{S+P}$ is known; P being Jupiter's sidereal period.

329. Cor. From the above the mean motion in longitude (m) of a satellite round Jupiter may be found; and if l be the Jovicentric longitude of a satellite at the time when the satellite suffers an occultation from Jupiter, the longitude at any time t from this $= l + mt$. But when the satellite suffers an occultation by Jupiter, its jovicentric longitude = the geocentric longitude of Jupiter, which can be determined; whence the jovicentric longitude of a satellite at any time is known. This supposes, indeed, that the motion in longitude of the satellite is constant, which is very nearly the case, since the orbits are nearly or quite circular.

330. If l, l', l'' be the jovicentric longitudes of the three first satellites at any time, it is found that $l - 3l' + 2l'' = 180^0$ very nearly. From this curious result it follows that the three first satellites cannot be eclipsed at the same time; for if the three satellites were eclipsed together, we ought to have at this time $l = l' = l''$; but if

$$l = l',\ l'' - l' = 90^0, \text{ and if } l = l'',\ l - l' = 60^0.$$

$$\text{Since } l - 3l' + 2l'' = 180^0;$$

$$\therefore\ dl - 3dl' + 2dl'' = 0,$$

where dl, dl', dl'' represent the motions of the three satellites.

331. *To find Jupiter's distance, from an observed eclipse of a satellite.*

Let S (fig. 63.) be the Sun, I Jupiter, O the satellite, and E the Earth at the time of an eclipse, which, being known, the jovicentric longitude (l) of the satellite is known.

Let the Sun's longitude, $SE\text{♈} = \odot$.

$nI\text{♈}$ or $IE\text{♈}$ the geocentric longitude of Jupiter $= \lambda$, $SI = r$, $SE = R$.

$$\text{Then } SIE = nIO = l - \lambda,$$

$$IES = \quad = \lambda - \odot;$$

$$\therefore\ r : R :: \sin(\lambda - \odot) : \sin(l - \lambda);$$

$$\therefore\ r = \frac{\sin(\lambda - \odot)}{R \,.\, \sin(l - \lambda)}.$$

332. If an eclipse of a satellite be observed when Jupiter is nearly in opposition with the Sun, and from the known synodic motion of the satellite, the following eclipses be calculated, they are observed to happen always later than the calculated time, and the difference goes on increasing, till Jupiter is in conjunction with the Sun, when it attains its maximum 16′ 26″. From this Römer conjectured, that the eclipses were not seen by us at the instant of happening, but that light takes 8′ 13″ to pass over a radius of the Earth's orbit. This hypothesis fully explains all the appearances, and also accounts for the change of place in the fixed stars treated of in the chapter on the aberration of light.

333. *To find the line of nodes of the orbits of Jupiter and one of its satellites.*

When the duration of an eclipse of a satellite is greatest, the satellite must pass through the middle of Jupiter's shadow, which lies in the plane of Jupiter's orbit; therefore, the satellite must be in or very near its node at the time of opposition, and the heliocentric longitude of the node must be the same as that of Jupiter; this latter is known from the time of the observation; therefore, the direction of the line of nodes is determined. The method is not applicable to the two first satellites, the duration of whose eclipses cannot be observed; but if the immersion be observed, and the time of the satellites' opposition be taken for the middle of the eclipse, half the time of an eclipse may be determined, and observing when this is greatest, the rest of the operation will be the same as before.

334. *To find the inclination of a satellite's orbit to the plane of the orbit of Jupiter.*

Let DCE (fig. 64.) be a section of Jupiter's shadow at the distance of the satellite, made when the duration of the eclipse is near its minimum, DCN the path of the satellite intersecting Jupiter's orbit in N, draw AB perpendicular to DC, which may be considered as a right line, and let t be the duration of half the eclipse, and T of the longest eclipse; then

$$BC : AE :: t : T \text{ nearly};$$

$$\therefore\ BC = \frac{t}{T} . AE,$$

$$\therefore\ AB = AE . \sqrt{1 - \frac{t^2}{T^2}}.$$

But $AC : 360^0 :: T : \sigma$,

σ being the synodic time of the satellite;

$$\therefore\ AB = \frac{T}{\sigma} . 360^0 . \sqrt{1 - \frac{t^2}{T^2}},$$

and $AN =$ the longitude of the node − the heliocentric longitude of Jupiter $= n - l$; therefore, from the spherical triangle ABN,

$$\sin 360^0 \frac{\sqrt{T^2 - t^2}}{\sigma} = \sin (n - l) . \sin N;$$

$$\therefore\ \sin N = \frac{\sin \frac{\sqrt{T^2 - t^2}}{\sigma} . 360^0}{\sin (n - l)}.$$

The above method is not quite accurate, since Jupiter is much depressed at its pole, and the minor axis, which is nearly perpendicular to the ecliptic, is to the major :: 13 : 14. In consequence of this, the section DCE, seen from the Sun, is not circular, and an inclination more near the truth will be obtained by diminishing the value of N, found above in the ratio of 13 : 14.

335. When a satellite is at its greatest apparent distance from Jupiter, its apparent distance from Jupiter's center : the

apparent equatoreal radius of Jupiter :: the true distance from Jupiter : true radius of Jupiter. If, therefore, the apparent distances from Jupiter, and Jupiter's equatoreal radius be measured at the same time with a micrometer, the distances of the satellites may be found in parts of Jupiter's radius. These distances are, calling Jupiter's equatoreal radius unity,

6.0485, 9.6235, 15.3502, 26.9983.

And the corresponding sidereal times are

$1^d.7691$, $3^d.5511$, $7^d.1545$, $16^d.6887$,

between which, the proportionality of the squares of the periods to the cubes of the mean distances may be observed.

Jupiter has also several dark stripes parallel to his equator, which are called his Belts; they often change their appearance, and seem to resemble clouds.

336. Saturn has seven satellites, all of which, except the sixth, cannot be seen without powerful telescopes; their distances, expressed in parts of Saturn's equatoreal radius, and sidereal periods, are

Distances.	Sidereal Period.
3.35	$0^d.943$,
4.30	1.370,
5.28	1.888,
6.82	2.739,
9.58	4.517,
22.08	15.945,
64.36	79.330.

The Ring of Saturn.

337. Saturn is also accompanied by a broad circular ring of no considerable thickness, inclined at an angle of 30^0 to the ecliptic, in the center of which Saturn seems suspended without touching the inner boundary of the ring. The ring is illuminated by the Sun and is invisible to us, when its plane passes through the Sun, as only the edge is then illumined, or when its plane passes through the Earth when the edge alone is seen, or lastly, when the dark side of the ring, upon which none of the Sun's rays fall, is turned to us.

Since the plane of the ring is not perpendicular to the line joining the centers of the Earth and Saturn, the two circular boundaries will appear to be two similar ellipses, and the ratio of the axes may be determined in a given position of the ring.

Take Saturn's center (fig. 65.) for the center of the sphere, and let the plane parallel to the ecliptic cut it in the great circle ΥNR, Υ being the first point of Aries, and let the ring cut the sphere in the arc NA; let T be the Earth seen from Saturn on the sphere, and draw AT, RT, perpendicular to NA, ΥR, respectively. Then TA measures the angle at which the line joining the centers of the Earth and ring is inclined to the plane of the ring, and since the ring will appear projected on a plane perpendicular to this line, $90^0 - TA$ is the inclination of the plane of the ring to the plane of projection; therefore, if $2a$ be the diameter of the ring or major axis of the projection, and $2b$ the minor axis,

$$b = a . \cos(90^0 - TA) = a . \sin TA.$$

Let λ be the geocentric longitude of Saturn's center; therefore, ΥR, the Saturcentric longitude of the Earth, $= 180^\circ + \lambda$, let ΥN, the longitude of the ring's node, $= n$;

$$\therefore NR = 180^\circ + \lambda - n.$$

Let ANR, the inclination of the ring to the ecliptic, $= \nu$. Then, since the ecliptic has been supposed to pass through Saturn's center, if β be the geocentric latitude of Saturn, TR will equal $-\beta$. Hence, knowing NR, RT, and the angle ANR, AT may be found in the same manner as the latitude from the given right ascension, declination, and obliquity. Thus we obtain

$$\frac{b}{a} = \sin AT$$

$$= \sin \nu \,.\, \cos \beta \,.\, \sin (n - \lambda) + \sin \beta \,.\, \cos \nu.$$

338. Six satellites have been discovered attending Uranus, their distance from which, in parts of the radius of Uranus, and their sidereal periods, are

Distances.	Sidereal Periods.
13.12	$5^d.81$,
17.2	8.71,
19.84	10.96,
22.75	13.46,
45.51	38.07,
91.01	107.69.

From the difficulty of observing them, little is known of the other elements of these satellites, or of those of Saturn. They observe like those of Jupiter and Saturn the proportionality of the squares of the periods to the cubes of the mean distances.

339. If a remarkable spot, observed on the disk of any planet or satellite at different times, has changed its apparent place more than naturally arises from the combined motions of the Earth and planet, the planet must have revolved round an axis, and the time of rotation may be determined from this circumstance. The times of rotation of the Sun, Mercury, Venus, Mars, Jupiter, Saturn, have been determined, as well

as those of Jupiter's satellites and the seventh satellite of Saturn, and the strongest analogy induces us to believe that the rotations of all the other bodies of the system have not been discovered, only from the difficulty of observing them. All these rotations are as the other planetary motions from west to east, and it is remarkable that the time of rotation of any satellite = the time of revolution round its primary.

The Sun's rotation is determined from observations made on dark spots which appear often on the Sun's disk, and vanish after some time. They frequently disappear soon after being formed, and at other times continue long enough to be brought into sight on the west side of the Sun's disk, and carried out on the east several times before they finally vanish. The mean interval between two appearances of the same spot is about 27 days, and as this is the synodic time of the Earth and spot, this gives a period of about 25 days for the time of the Sun's rotation.

340. *To find the heliocentric latitude and longitude of a spot on the Sun's disk.*

Let S (fig. 66.) be the Sun's center, E the Earth's, P the spot referred to the ecliptic by the perpendicular PN.

Let $l=$ the heliocentric longitude of the Earth,

$x=$ ....spot;

$\therefore ESN=l-x$.

Let PSN the heliocentric latitude $=y$,

PEN the geocentric latitude $=\beta$,

SEN the difference of geocentric long. of S and $P=E$,

$\Delta=$ the Sun's semi-diameter.

Then $SP.\sin y=PN=EP.\sin\beta=SE.\sin\beta$, nearly;

$$\therefore\ \sin y=\frac{SE}{SP}.\sin\beta=\frac{\sin\beta}{\sin\Delta}.$$

$$\text{and } SP.\cos y : EP.\cos\beta :: SN : NE$$

$$:: \sin E : \sin(l-x);$$

$$\therefore \sin(l-x) = \frac{\sin E.\cos\beta}{\cos y}.\frac{EP}{SP}$$

$$= \frac{\sin E.\cos\beta}{\sin\Delta.\cos y}$$

$$= \frac{\sin E.\cos\beta}{\sqrt{\sin^2\Delta-\sin^2\beta}}$$

$$= \frac{\sin E.\cos\beta}{\sqrt{\sin(\Delta+\beta).\sin(\Delta-\beta)}}.$$

341. *To find the inclination of the Sun's equator to the ecliptic, and the time of the Sun's rotation.*

Suppose E (fig. 67.) the pole of the ecliptic, P of the Sun's equator, A, A', A'' heliocentric places of the same spot, observed at three different times; join EA, EA', EA'', PA, PA', PA''; then EA, EA', EA'', the co-latitudes, and AEA', AEA'', $A'EA''$, the differences of longitude may be determined by the last article from geocentric observations; therefore, in each of the triangles AEA', AEA'', $A'EA''$, two sides and the included angle are given, from which the bases, and the angles at the bases, may be determined; therefore, the angles $A'AA''$, $AA'A''$, $AA''A'$, which are the sums of two angles at the bases, may be found. Now, because P is the pole of the Sun's equator, parallel to which the spot revolves,

$$PA = PA' = PA'',$$

and if PQ, PR are respectively perpendicular to AA', AA'',

$$AQ = \tfrac{1}{2}(AA'), \text{ and } AR = \tfrac{1}{2}AA''.$$

Therefore, by Napier's rules,

$$\cos PAA' = \tan AQ.\cot AP \dots\dots\dots (1),$$

$$\cos PAA'' = \tan AR.\cot AP \dots\dots\dots (2);$$

$$\therefore \cos PAA' : \cos PAA'' :: \tan AQ : \tan AR;$$

$$\therefore \cos PAA' + \cos PAA'' : \cos PAA' - \cos PAA''$$

$$:: \tan AQ + \tan AR : \tan AQ - \tan AR,$$

$$\text{or } \cot \tfrac{1}{2} . A'AA'' : \tan \tfrac{1}{2} . (PAA'' - PAA')$$

$$:: \sin (AQ + AR) : \sin (AQ - AR);$$

$$\therefore \tan \tfrac{1}{2} . (PAA'' - PAA') = \frac{\sin (AQ - AR)}{\sin (AQ + AR)} . \cot \tfrac{1}{2} . A'AA''.$$

If $\frac{1}{2} . (PAA'' - PAA')$, determined by this equation, be subtracted from $\frac{1}{2} (PAA' + PAA')$, PAA' is determined, and, therefore, $PAE = PAA' - EAA'$; also, AE is known, and PA determined by equation (1); therefore, in the triangle, PAE, two sides PA, and AE, and the included angle PAE, are known, from whence PE, the inclination of the Sun's equator to the ecliptic, may be found. Also from the triangle APQ we may find the angle APQ; and therefore its double APA'; and $APA' : 360^0$ :: the observed time from A to A' : the time of the Sun's rotation.

CHAP. XVI.

ON ECLIPSES, AND OCCULTATIONS OF FIXED STARS BY THE MOON.

342. SINCE the Earth and Moon are round opaque bodies, they intercept the Sun's rays, and form behind them dark conical shadows. When the shadow thus formed by the Earth falls upon the Moon's disk, which can happen only at the time of full Moon, or when the Moon and Sun are in opposition, the Moon is said to be *eclipsed;* and the eclipse is called *total* or *partial,* according as the whole or part of the Moon's surface is deprived of the Sun's light. Round the conical shadow which is produced by the Earth intercepting the whole of the Sun's light, another, called the *penumbra,* is formed by some parts of the Earth intercepting part of the Sun's light; so that during a lunar eclipse, the Moon enters the penumbra first, and the obscuration on the Moon's surface increases, till it enters the true shadow, and becomes eclipsed.

The new Moon produces effects on the Earth, similar to those which the Earth produces on the full Moon, for the Moon being then in conjunction with the Sun, may prevent all, or parts of the Sun's light, from reaching places on the Earth's surface, and thus cause total or partial eclipses of the Sun.

343. The cause of lunar eclipses not happening, every full Moon, or solar eclipses every new Moon, exists in the Moon's orbit not coinciding with the ecliptic, but being inclined to it at an angle of 5° 9′; for if the Moon's latitude, when in opposition, exceeds the semi-diameter of a section of the Earth's shadow at

the distance of the Moon + the semi-diameter of the Moon, the Earth's shadow will fall above or below the Moon, and there will be no lunar eclipse. For the same reason the shadow of the new Moon may fall above or below the Earth; so that eclipses can only happen when the Moon is near its node at the time of syzygy.

344. The synodic period of the Sun and Moon is $29^d.53058$, and that of the Sun and the node of the Moon's orbit, is $346^d.6196$, which numbers are to each other as 19 : 223, nearly; if, therefore, the Moon and node are both in syzygy at the same time, they will be together again under similar circumstances after 223 lunations, or 19 revolutions of the node, and eclipses will recur again in the same order after an interval of 223 lunations, or $18^y\ 11^d$, nearly.

If the ratio of 223 : 19 were exact, and the Sun and Moon moved uniformly, nothing more would be requisite for the calculation of future eclipses, than the registered observations of eclipses during $18^y\ 11^d$; but as both these hypotheses are incorrect, the method can only be regarded as an approximation which may be used as a guide to more exact investigations.

345. *To find the length of the Earth's shadow, and the sections of the umbra and penumbra, at the distance of the Moon.*

Let ABC, abc (fig. 68.) be sections of the Sun and Moon, made by a plane passing through their centres; BD, CD common tangents meeting in D; BOc, COb common tangents meeting in O; then if the figure revolve round SD, BDC will generate the conical shadow of the Earth into which none of the Sun's rays enter, and mbp, ncq the penumbra which admits only part of the Sun's rays. Let $pmnq$ be a section of these figures, made a plane perpendicular to SO at the distance of the Moon, and let p be the Moon's horizontal parallax, π the Sun's, μ and σ their apparent semi-diameters, and R the radius of the Earth; then

$$EDB = SEB - EBD = \sigma - \pi;$$

$$\therefore\ ED = \frac{Eb}{\sin EDb} = \frac{R}{\sin(\sigma - \pi)}.$$

The radius of a section of the shadow at the Moon

$$\begin{aligned} &= MEm \\ &= EmB - EDm \\ &= p + \pi - \sigma. \end{aligned}$$

The radius of a section of the penumbra

$$\begin{aligned} &= MEp \\ &= EOp + EpO. \end{aligned}$$

$$\begin{aligned} \text{But } EOp &= SOB \\ &= OBD + ODB \\ &= 2\pi + \sigma - \pi = \pi + \sigma; \end{aligned}$$

$$\therefore\ MEp = p + \pi + \sigma.$$

346. Cor. The numerical values of ED, MEm, MEp are subject to change, from variations in the distances of the Sun and Moon from the Earth,

ED varies from 212 to 220 radii of the Earth.

MEm .... 37′ 42″ ..... 45′ 52″,

MEp .... 1° 8′ 12″.... 1° 18′ 27″.

It was remarked, that no eclipse can take place when the distance of the Sun from the node of the lunar orbit at the time of the Moon's being in syzygy, exceeds a certain quantity, this quantity, which is not the same in lunar and solar eclipses, is called, in the former case, the *lunar ecliptic limit*, and in the latter the *solar ecliptic limit*.

347. *To find the lunar ecliptic limit.*

Let ON (fig. 69.) be the ecliptic, N the node of the Moon's orbit LN, O the center of the Earth's shadow when the Moon is in opposition, S the center of the shadow when the Moon is in the node; therefore in the time from L to N, the Moon approaches S by the difference of latitude LO, and recedes from it by the difference of longitude SN. Take $SM=LO$, and join MN; then, considering LN, ON, MN to be right lines, if a body describe MN uniformly in the time of the Moon's moving from L to N, it will approach and recede from the fixed point S in the same manner as the Moon in its orbit LN does from S in motion; if, therefore, Sm be drawn perpendicular to MN, Sm is the nearest approach of the centers of the Moon and shadow, and taking $Sm\,(\Delta)$ = the greatest semi-diameter of the Moon $10'\ 45''$ + the greatest semi-diameter of the shadow at the distance of the Moon $45'\ 52''$, the greatest value of SN or ON, at which an eclipse is possible, will be obtained. To find SN; let m be the Moon's horary motion in longitude, s the Sun's, which is the same as that of the shadow, $f=m-s$, the relative horary motion of the Moon and shadow, g the Moon's horary motion in latitude. Then, because SM, SN are described in the same time with the uniform motions g and f,

$$f : g :: SN : SM :: 1 : \tan N = \frac{g}{f};$$

$$\therefore\ SN = \frac{\Delta}{\sin N} = \Delta\frac{\sqrt{f^2+g^2}}{g}.$$

$$\text{Now } ON : SN :: m : f;$$

$$\therefore\ ON = \Delta \cdot \frac{m}{f} \cdot \frac{\sqrt{f^2+g^2}}{g}.$$

The numerical value of ON is $12^\circ\ 36'$; if, therefore, the difference of longitudes of the Moon and node at the time of opposition be greater than $12^\circ\ 36'$, no lunar eclipse can take place, if $ON = 12^\circ\ 36'$ the Moon's disk may just touch the shadow, and if ON is less than $12^\circ\ 36'$, there may be a lunar eclipse.

Similarly, if Δ' be taken equal, the least semi-diameter of the Moon 14′ 40″ + the least semi-diameter of the shadow 37′ 42″,

$$ON' = \Delta' . \frac{m}{f} . \frac{\sqrt{f^2+g^2}}{g} = 9^0,$$

the least ecliptic limit is obtained; if, therefore, the Moon at the time of opposition be within 9^0 of the node, there must be a lunar eclipse. Between the distances 9^0 and 12^0 36′, an eclipse is doubtful, and the exact values of the radii of the Moon and shadow must be found in order to ascertain whether there will be a lunar eclipse or not.

348. In lunar eclipses the Earth's atmosphere deflects and absorbs those rays of the Sun, which pass near the Earth's surface, and thus causes the shadow of the Earth to appear somewhat larger than it would otherwise be. In order to take this effect into account, astronomers consider the radius of the shadow to be one-sixtieth greater than the value found in Art. 345, or add as many seconds as there are minutes in the radius of the section. Some of the rays deflected by the atmosphere enter into the shadow, and cause a faint light sometimes perceived on the Moon's disk in the middle of a lunar eclipse.

Another singular effect of our atmosphere has been sometimes observed, by which the Sun and Moon were both so elevated by refraction, as to be visible above the horizon during a total eclipse of the Moon.

349. *To find the time, magnitude, and duration of a lunar eclipse.*

Let ABB' (fig. 70.) be a section of the Earth's shadow at the distance of the Moon, Sn the path described by its center S on the ecliptic; Mn the relative orbit of the Moon, Mn, Sn being considered straight lines, as in Art. 347. Draw SO perpendicular to Sn and Sm to Mn, then O and m are in the places, with respect to S, of the Moon in opposition, and at the middle of the eclipse.

Let $a = SB = p + \pi - \sigma$ the radius of the section of the shadow,

$\beta = SO$, the Moon's latitude in opposition,

$f =$ the relative horary motion in longitude of the Sun and Moon, or the motion in longitude of the Moon in the relative orbit Mn,

$h =$ the Moon's horary motion in the relative orbit,

$g =$ .... latitude;

$\mu =$ .... semi-diameter;

$$\therefore \tan n = \frac{g}{f}, \text{ and } g = h \sin n.$$

Let M and M' be the places of the Moon's center at the time of the first and last contact; therefore

$$SM = SM' = a + \mu.$$

Now $Sm = \lambda \cos n$;

$$\therefore Mm = \sqrt{(a+\mu)^2 - \lambda^2 \cos^2 n} = M'm,$$

and $mO = \lambda \sin n$.

If, therefore, t and t' be the times from opposition of the first and last contact, and T the time of opposition,

$$t = \frac{Mm - Om}{h} = \left\{\sqrt{(a + \mu^2) - \lambda^2 \cos^2 n} - \lambda \sin n\right\} . \frac{\sin n}{g},$$

$$t' = \frac{M'm + Om}{h} = \left\{\sqrt{(a + \mu)^2 - \lambda^2 \cos^2 n} + \lambda \sin n\right\} . \frac{\sin n}{g};$$

$$\therefore \text{ the } \mathit{duration} = 2\sqrt{(a+\mu)^2 - \lambda^2 \cos^2 n} . \frac{\sin n}{g}.$$

The time from opposition of the middle of the eclipse

$$= \frac{Om}{h} = \frac{\lambda \sin^2 n}{g}.$$

The *magnitude* of the eclipse, or the part of the Moon immersed

$$= Su - Sv,$$

$$= Su - Sm + mv$$

$$= a - \lambda \cos n + \mu.$$

The Moon's diameter is generally divided into twelve equal parts called digits;

therefore the digits eclipsed : 12 :: $a - \lambda \cos n + \mu : 2\mu$

$$= \frac{6}{\mu} . (a - \lambda \cos n + \mu).$$

350. Cor. 1. If $\lambda \cos n$ be greater than $a + \mu$, t and t' are impossible, and no eclipse can take place, as is also evident from the figure.

351. Cor. 2. In exactly the same manner it may be proved, if t and t' be the times from opposition of the centers of the shadow and Moon being at any given distance c,

$$t = (\sqrt{c^2 - \lambda^2 \cos^2 n} - \lambda \sin n) . \frac{\sin n}{g},$$

$$t' = (\sqrt{c^2 - \lambda^2 \cos^2 n} + \lambda \sin n) \frac{\sin n}{g}.$$

352. Cor. 3. If $c = p + \pi + \sigma + \mu =$ the radius of the penumbra + the radius of the Moon, the times of the Moon entering and emerging from the penumbra are obtained.

353. The horary motion of the Moon is about $32\frac{1}{2}'$, and that of the Sun $2\frac{1}{2}'$; therefore the relative horary motion of the Moon is $30'$, and as the greatest diameter of the section at the distance of the Moon is $1^0\ 31'\ 44''$, a lunar eclipse may last more than 3 hours.

354. *To find at what place on the Earth's surface a given lunar eclipse is visible.*

Suppose t the Greenwich time of the beginning of the eclipse, and δ the Moon's declination, then, since the Moon is in opposition, a place with a latitude δ, and east longitude from Greenwich $(12-t).15^0$ will have the Moon in its zenith at the beginning of the eclipse; if, therefore, about this point as a pole a great circle be described on a common terrestrial globe, all places included between this circle and its pole, will see the beginning of the eclipse. In the same manner the circle bounding all places which see the end of the eclipse may be found; and those parts of the Earth which are common to the two divisions, have the Moon above the horizon during the whole time of the eclipse.

Solar Eclipses.

355. A solar eclipse is the same in its physical circumstances as an eclipse of the Moon, and the preceding formulæ may be changed, so as to apply to it. The objects of calculation are however very different in lunar and solar eclipses, being, in the former, to determine merely whether any part of the Sun's light is intercepted, and in the latter, not only this, but also from what part of the Earth it is intercepted. On this account eclipses of the Sun are treated somewhat differently from those of the Moon.

356. *To find the angular distance of the Moon and Sun's centers at the beginning of a solar eclipse.*

Let RB, rb, MO (fig. 71.) be sections of the Sun, Earth and Moon, made by a plane passing through their centers, Rr, Bb common tangents to the circles RB, rb. Then when the Moon enters any part of the luminous cone formed round the Sun and Earth, as at O, it intercepts a portion of the Sun's light from reaching the Earth, and the solar eclipse begins; therefore the distance of S and M, at the beginning of the eclipse

$$\begin{aligned}
&=SEM\\
&=SER+OER+OEM\\
&=SER+EOr-ERr+OEM\\
&=\sigma+\mu+p-\pi,
\end{aligned}$$

SEO the radius of the luminous cone at the distance of the Moon

$$=SEM-\mu$$

$$=\sigma+p-\pi.$$

357. From the figure, it is evident that the eclipse begins at *r* some time before it is visible at the center of the Earth, on account of the spectator at *r* being so much raised above *E*, or which is the same thing, on account of the Moon being so much depressed by parallax to a spectator at *r*. For the same reason, when the Moon appears emerging to a spectator at *E*, it may be so depressed by parallax at *r*, as to have emerged some time, whilst from the same cause, to a spectator at *b*, the emersion may not have commenced. The Sun is affected by parallax as well as the Moon, although in a much less degree, on account of the nearness of the Moon to the Earth; therefore the centers of the Sun and Moon are made to approach or separate by the difference of the lunar and solar parallax.

358. If tangents to the Moon's disk be drawn from any point of the Earth's surface as *r* at the instant of greatest obscuration, the eclipse at *r* will be *total*, when these tangents include the Sun, *partial* when they include part of it, and *annular* when they both fall within the Sun's disk.

359. The solar ecliptic limits may be found in the same manner as those of the Moon were, by making *Sm* (fig. 69.) = the greatest semi-diameter of the section of the luminous cone at the Moon + the greatest semi-diameter of the Moon, for the greatest limit; or, = the least semi-diameter of the section + the least radius of the Moon, for the least limit. Hence it appears, that if the distance of the Moon, in conjunction, from the node is

less than 13° 42′ the eclipse is certain,
between 13° 42′ and 19° ....possible,
greater than 19° ....impossible.

360. *To find the time and duration of a solar eclipse.*

Adopting the same figure and notation as in the lunar eclipse, with the exception of making $EB(a)$ the radius of a section of the luminous cone at the Moon $= \sigma + p - \pi$.

The time from opposition of the first contact

$$= \left\{\sqrt{(a+\mu)^2 - \lambda^2 . \cos^2 n} - \lambda \sin n\right\} \frac{\sin n}{g}.$$

The time from opposition of the last contact

$$= \left\{\sqrt{(a+\mu)^2 - \lambda^2 \cos^2 n} + \lambda \sin n\right\} . \frac{\sin . n}{g}.$$

The duration

$$= 2 \frac{\sin n}{g} \sqrt{(a+\mu)^2 - \lambda^2 \cos^2 n}.$$

The duration thus determined is not the duration of the eclipse seen from the centre of the Earth, or from any given point of the surface, but the whole duration, from the time of the eclipse being first visible, to the time when the Moon ceases to intercept any part of the Sun's light.

361. *To find the apparent difference of longitude and latitude of the Sun and Moon during a solar eclipse.*

Let n be the altitude, and N the longitude of the nonagesimal, at any time during the eclipse; l and λ the longitude and latitude of the Moon, and l' the longitude of the Sun.

Then the Moon's apparent longitude

$$= l - p \frac{\sin n . \sin (N - l)}{\cos \lambda}$$

$$= l - p \sin n . \sin (N - l'),$$

very nearly, because the Moon's latitude during a solar eclipse is very small, and the longitude nearly the same as the Sun's.

Similarly, the Sun's apparent longitude

$$= l' - \pi \sin n . \sin (N - l');$$

therefore the apparent difference of longitude

$$= l - l' - (p - \pi) . \sin n . \sin (N - l').$$

Again, the Moon's apparent latitude

$$= \lambda - p . \{\cos n . \cos \lambda - \sin n \sin \lambda \cos (N - l)\}$$

$$= \lambda - p \cos n, \text{ nearly,}$$

and the Sun's apparent latitude

$$= - \pi \cos n;$$

therefore, the apparent difference of latitude

$$\lambda - (p - \pi) \cos n.$$

362. *To calculate the circumstances of a solar eclipse at a given place.*

Having found, as in Art. 360, the times T and T' of the beginning and end of the whole eclipse; the *true* longitudes of the Sun and Moon, and the true latitude of the Moon corresponding to those times, may be taken out of the solar and lunar tables; and correcting the values thus found for parallax, the apparent differences of longitude of the Sun and Moon may be found. Let l, l' be the apparent differences of longitude, and λ and λ' of latitude for the times T and T'; then if τ be the time of apparent opposition, since the Sun and Moon approach in longitude by $l - l'$ in the time $T' - T$;

$$\therefore l - l' : l :: T' - T : \tau - T;$$

$$\therefore \tau = T + \frac{l}{l - l'} . (T' - T).$$

Also $\frac{l - l'}{T' - T}$ and $\frac{\lambda' - \lambda}{T' - T}$ are the apparent horary approaches of the Sun and Moon in latitude and longitude; therefore using these values instead of f and g, and the apparent difference of latitude for the time τ instead of λ, all the circumstances of a solar eclipse at a given place may be calculated in the same manner as the lunar eclipse, Art. 349.

363. The circumstances of a solar eclipse have been calculated in Articles 360, 362, upon the supposition of the Sun and Moon moving uniformly in right lines during the whole time of the eclipse. This hypothesis was sufficiently accurate for the calculation of the circumstances of a lunar eclipse, in which the times of the beginning and end cannot be exactly observed on account of the penumbra, but in a solar eclipse, where the times of first and last contact can be observed with considerable accuracy, a more exact computation becomes necessary. The time when a solar eclipse commences at a given place, may be found with great accuracy in the following manner.

Second method of calculating the time of a solar eclipse.

364. Let T be the approximate time of the beginning of the eclipse found by Art. 362. $T+t$ the true time; for the time T, let the apparent difference of longitude of the Sun and Moon (l), and the apparent latitude of the Moon (λ) be found (Art. 361), and also f and g, the horary increments of these quantities, which may be deduced by subtracting the values of l and λ, calculated for half an hour before T, from those calculated half an hour after; therefore $l+ft$, $\lambda+gt$ are the values of l and λ at the beginning of the eclipse. Let S (fig. 72.) be the center of the Sun, M of the Moon at the beginning of the eclipse, and P the pole of the ecliptic SN; then

$$SN = l+ft, \quad MN = \lambda+gt;$$

and if μ and σ are the apparent diameters of the Sun and Moon

$$SM = \sigma+\mu;$$

therefore by the right-angled triangle SMN, which may be considered rectilinear, since SN, MN, SM are very small

$$(\sigma+\mu)^2 = (l+ft)^2 + (\lambda+gt)^2 \ldots\ldots\ldots\ldots (a),$$

from which quadratic t may be found.

The least value of t must be taken, since, by hypothesis, t is very small; the other value belongs to the time of last contact, which may be determined from it, but not so accurately as if the values of l, λ, f, g are calculated for the approximate time of the last contact, and the true time found by an equation similar to (a).

365. Cor. 1. If t be very small, or the approximate value T be very nearly the time of contact, as generally happens, the square of t may be neglected, and t may be determined from the simple equation

$$(\sigma+\mu)^2 = l^2+\lambda^2+2\,(f+g)\,t;$$

$$\therefore\ t = \frac{(\sigma+\mu)^2 - l^2 - \lambda^2}{2\,.(f+g)}.$$

366. Cor. 2. If μ be affected with the negative sign, the same equation determines the time of internal contact, when the Moon's disk has just entered within that of the Sun.

367. If the Sun and Moon be referred to the equator, and if δ be the Sun's declination at the approximate time of contact T, δ' and a' the differences of apparent declination of the Sun and Moon for the same time, (which may be deduced by correcting the difference of the true declinations and right ascensions for parallax, and f and g the horary increments of δ' and a'; then if M (fig. 72.) be the Moon, S the Sun, and P the pole of the *equator* at the true time of contact $T+t$,

$$SM = \mu+\sigma,\ \ MN = \delta'+gt,\ \ SN = (a'+ft)\,.\cos\delta;$$

$$\therefore\ (\mu+\sigma)^2 = (\delta'+gt)^2 + (a'+ft)^2.\cos\delta.$$

From which equation, deducing the two values of t, t and t', $T+t$ and $T+t'$ the times of the first and last contact, are obtained as before.

368. *To construct for the places on the Earth's surface where a given solar eclipse is visible.*

The circumstances of an eclipse, which depend only on the distance of the Sun and Moon, will not be altered if the Sun be considered unaffected by parallax, and the Moon depressed to or from it, by the whole difference of solar and lunar parallax. Let S and M (fig. 73.) be the true centers of the Sun and Moon seen from the center of the Earth at any given time during the eclipse. With S as pole, and $Sv = \mu + \sigma - n$ (n expressed in digits of the Sun's disk), describe a small circle round S, take Mv any quantity less than $p - \pi$, and produce the arc vM to a point Z, such that

$$Mv = \overline{p - \pi} \,.\, \sin ZM.$$

Then, at a place the zenith of which is Z, M will be depressed by parallax through Mv, and its apparent distance from S will be

$$Sv = \mu + \sigma - n;$$

or n digits of the Sun will appear eclipsed. In the same way it may be proved that n digits will appear eclipsed at a place, the zenith of which is Z', $Z'v'$ being symmetrical to Zv on the other side of SM. By taking other points in the arc vv', other places may be determined, in which n digits appear eclipsed, and if T, T' be the two extreme points, determined by taking

$$Mu = Mu' = p - \pi, \text{ and } TM = T'M = 90^0,$$

n digits will appear eclipsed at the same given instant through the whole curve TT'. If $n = 0$, the curve in which a contact just appears is determined; if

$$n = \mu + \sigma, \quad Sv = Su = 0,$$

and the eclipse is central; in this case, however, only one point is determined, since u and u' coincide. In the same manner, by giving to n different values, all the places which see the eclipse at the instant when the Sun and Moon have the true distance SM may be found; and by altering the value of SM and repeating the same process, all places at which the eclipse is visible may be determined.

When the eclipse is first visible,

$$SM = \mu + \sigma + p - \pi,$$

in this case also u and u' coincide, and one place only is determined, at which the eclipse is seen when the Sun and Moon are in the horizon.

By determining in this manner a few of the extreme places at which a given eclipse is seen, the intermediate places at which the eclipse is visible, may be easily found on a common globe.

369. *To calculate from the preceding construction the places where a given solar eclipse is visible.*

Through the pole of the equator P (fig. 73.) draw the declination circles PZ, PM, PG, PS, of which PG passes through the zenith of Greenwich; therefore ZPG is the terrestrial longitude of the place whose zenith is Z. Let $MS = c$, $SMv = \phi$, $SMP = \theta$, $Mv = P$, $\tau =$ the time at Greenwich, when the true distance of the Sun and Moon's center is c, calculated by Art. 360, $\Delta = PM$, the Moon's north polar distance, and $\Delta' = PS$, the Sun's. Then

$$\cos\phi = \frac{\cos(\mu + \sigma - n) - \cos c \, . \cos P}{\sin c \, . \sin P},$$

$$\cos\theta = \frac{\cos\Delta' - \cos\Delta \, . \cos c}{\sin\Delta \, . \sin c},$$

$$PMZ = 180^0 - (\theta + \phi),$$

$$\sin ZM = \frac{P}{p - \pi},$$

$$\cos PZ = \sin ZM . \sin\Delta . \cos(180^0 - \theta - \phi) + \cos ZM . \cos\Delta$$

which determines the terrestrial latitude of Z,

$$\text{and } \cos ZPM = \frac{\cos ZM - \cos PZ \, . \cos\Delta}{\sin PZ \, . \sin\Delta},$$

$$\cos MPS = \frac{\cos c - \cos\Delta \, . \cos\Delta'}{\sin\Delta \, . \sin\Delta},$$

$$ZPS = ZPM + MPS;$$

therefore the terrestrial west longitude of the place whose zenith is Z

$$= ZPS - GPS$$

$$= ZPS - \text{the apparent time at Greenwich is known.}$$

In order to find all the places which see the different phases of the eclipse, P and c must be made to vary in the manner pointed out in Art. 368.

For actual calculation, the equations of this article may be simplified, and expedients adopted, by which the labour of finding many places where the eclipse is visible is much lessened. See Delambre, *Astronomie*, Chap. xxvi.

On the Number of Eclipses.

370. The Sun is more than a month in passing through the solar ecliptic limits; therefore there *must* necessarily be two eclipses of the Sun in a year, one at each node. If a solar eclipse happens very near the node; at the preceding conjunction the Sun had not entered the ecliptic limits, and at the succeeding conjunction will have passed beyond it; therefore there *may* be only one eclipse at each node, or two solar eclipses in a year. If a solar eclipse happen very soon after the Sun has entered the ecliptic limits, there may be another eclipse near the time when it leaves the ecliptic limits, and as the same may happen at each node, there *may* be four solar eclipses in a year.

The Moon is less than a month in passing through the lunar ecliptic limits; therefore there *can* be no more than one eclipse at each node, and there *may* be none; or there *can* be only two lunar eclipses in a year, and there *may* be none.

Hitherto the node has been considered stationary, but since the node recedes about 20^0 each year, if a solar or lunar eclipse happen near the beginning of the year, there may be another of the same kind, at the *same* node, near the end of the year; therefore upon the whole there may be seven eclipses; five of the Sun and two of the Moon, or four of the Sun and three of the Moon, in the course of a year.

On the Occultation of fixed Stars by the Moon.

371. When the Moon passes between a fixed star and the Earth, the star is said to suffer an *occultation*. From this

description it is obvious, that an occultation is very similar in its physical circumstances to a solar eclipse, but is simpler, on account of the star being without motion, and unaffected by parallax. As in a solar eclipse, the times of occultation will be different at different places, on account of parallax, and an occultation may happen at some places, and not at others.

372. *To find the time of an occultation.*

From the known longitudes of the Moon and star, and the Moon's motion in longitude at the time when its longitude = that of the star, the time (T) of conjunction of the Moon and star may be found. Let l and λ be the differences of longitude and latitude of the two bodies for the time T, f and g the horary increments of l and λ at the same time, and s the latitude of the star; then, as in Art. 364. if S (fig. 72.) be the star, M the Moon, and P the pole of the ecliptic at the time of first or last contact, since the star's diameter is 0,

$$SM=\mu,\ MN=\lambda+gt,\ NS=(l+ft)\,.\cos s;$$

$$\therefore\ \mu^2=(\lambda+gt)^2+(l+ft)^2.\cos^2 s,$$

from which equation finding t and t', the two values of t, we obtain $T+t$ and $T+t'$, the times of the beginning and end of the occultation.

If λ and l are the differences of the geocentric longitudes and latitudes of the Moon and star, $T+t$ and $T+t'$ are the times at which the occultation would be observed from the center of the Earth, but if λ and l are the differences of the true latitude and longitude of the star, and of the apparent latitude and longitude of the Moon corrected for parallax at a given place, the apparent times of the first and last contact at the place of observation are obtained.

CHAP. XVII.

ON THE TRANSIT OF VENUS OVER THE SUN'S DISK.

373. When Venus is near the node of her orbit, at the time of conjunction, she lies between the Sun and the Earth, and is seen to pass like a dark spot over the Sun's disk.

The transit of an inferior planet over the Sun's disk is exactly the same, mathematically considered, as a solar eclipse, and the same formulæ may be applied to calculate the one which were used for the other, by changing the quantities f, g, &c. which have reference to the Moon into the corresponding quantities for the planet; therefore, if g be the motion of Venus in latitude, f the relative motion in longitude of Venus and the Sun, λ the latitude of Venus at the time of conjunction (T), c the sum of the diameters of the Sun and Venus, and $\tan n = \frac{g}{f}$.

The time when the transit begins

$$= T + \frac{\sin n}{g} . \{\lambda \sin n - \sqrt{c^2 - \lambda^2 \cos^2 n}\}.$$

The time when the transit ends

$$= T + \frac{\sin n}{g} . \{\lambda \sin n + \sqrt{c^2 - \lambda^2 \cos^2 n}\}.$$

If $c =$ the sum of the semi-diameters of the Sun and Venus, the times of first and last contact, as seen from the center of the

Earth are determined, but if $c =$ the semi-diameter of the Sun — the diameter of Venus, the times of Venus having entered and of beginning to emerge from the Sun's disk, are obtained.

374. As in a solar eclipse, spectators on the Earth's surface refer the planet to different parts of the Sun's disk, on which account the observed times of the first and last contact, and the whole duration of the same transit, are different in different places. This difference arises from the difference of parallax of the Sun and Venus, and may be found by the formulæ of the preceding Chapter, when the difference of parallax is given; therefore, conversely, when the difference of the times of duration is known by observation, the difference between the two parallaxes may be obtained.

375. The ratio of the parallaxes of Venus and the Sun is known from the ratio of their distances; therefore, if the difference of the parallaxes be found from the observed durations of the same transit in different places, the Sun's parallax may be determined. On account of the accuracy with which the entrance of a dark body like Venus on the Sun's disk may be observed, the duration of a transit may be determined with great exactness, which renders this by far the best method of finding the Sun's parallax. We may judge of its exactness, by considering that the greatest difference in the durations of the transit of 1769, observed at different places, was at least 18′; therefore, as the Sun's parallax is about $8''.5$, it would require an error of $\frac{1}{8.5} . 18'$, or more than 2 minutes, in the difference of the observed times, to produce an error of $1''$ in the computed parallax of the Sun. As it is highly probable that the difference of the times of duration is known within, $15''$, therefore, it is in the same degree probable that the Sun's parallax was determined within $0''.25$ of its true value.

If the observed difference had been 25′, it would have required an error of nearly 3′ in the time of duration to produce an error of $1''$ in the computed solar parallax.

376. *To find what places are most eligible for observing the transit.*

Since the error in the computed parallax of the Sun diminishes as the difference of the observed times of transit increases, it becomes important to enquire at what places it is most eligible to make the observations. Not to enter into exact calculations, which may be made in a manner similar to those of Art. 369, we may determine the most eligible places with sufficient accuracy by means of a common globe.

Suppose (as was nearly the case in the transit of 1769) that the transit begins at 7 o'clock, apparent Greenwich time, and continues 6 hours, and that the Sun's declination at the same time is 22^0 north. Place the globe so that the altitude of the north pole may equal 22^0, and turn it till the places in west longitude $7.15^0 = 105^0$ are brought under the fixed meridian; then, in this position of the globe, the Sun will be visible at all places above the horizon of the globe, and will be in the zenith of a place in north latitude 22^0. If now the globe be turned through $6.15^0 = 90^0$, all places above the horizon in the second position will see the Sun at the time of last contact, and those places which have continued above the horizon whilst the globe was turned, will see the whole of the transit.

In the first position of the globe the commencement of the transit is not seen by all places in the horizon, but only at one place, which lies in the great circle passing through the centers of the Sun and Venus, which places may be determined as in Art. 368.

The place which has the Sun in its zenith at the beginning of the transit, sees the first contact nearly at the time when it would be seen from the center of the Earth; but as this place is near the horizon when the transit ends, the duration is increased. Since the latitude of Venus is north, at places in higher northern latitudes than 22^0, Venus is depressed by parallax towards the Sun's center, and the duration of the transit is prolonged, and this effect is increased with the north latitude of the observer. For the same reason, the duration is diminished at places in south latitude; therefore, among all accessible places which are above the horizon in the two positions, those ought to be chosen which are most north or south, in order that the observed difference of duration may be the greatest possible.

Places in high north latitudes, where the nights are short, may see the first contact before sun-set, and the last after sunrise, and in this case also, the duration of the transit would be increased.

377. *To find when a transit of Venus may be expected.*

A transit can only happen when Venus is near its node at the time of conjunction; if, therefore, the Earth and Venus are together in the same node at any time, when they return to the same position again, each of them must have made a number of complete revolutions. Let m be the number of revolutions made by Venus, n the number made by the Earth in the interval between two conjunctions; then, since the Earth revolves in $365^d.256$, and Venus in $224^d.7$, we must have

$$m\,.\,365^d.256 = n\,.\,224^d.7\,;$$

$$\therefore\ \frac{m}{n} = \frac{224.7}{365.256}.$$

The continued fractions approximating to $\frac{m}{n}$ are

$$\frac{8}{13},\quad \frac{235}{382},\quad \frac{713}{1159},\ \&c.$$

Therefore, a transit may be expected at the same node after an interval of 8 or 235 or 713 years. Now, a transit happened at the descending node in 1769, and at the ascending node in 1639; therefore, transits may be again expected at the descending node in

2004, 2012, 2247, 2255, 2490, 2498, 2733, and at the ascending node in

1874, 1882, 2117, 2125, 2360, 2368, 2603.

To ascertain whether there will be a transit at these times, a more exact investigation must be made, and the transit will happen or not, according as the least distance of the Sun and Venus is less or greater than the sum of their semi-diameters.

378. *To find the Sun's parallax from the difference of the times of duration of the same transit, observed at different places.*

Let T, T' be the *Greenwich* times when the first and last contact would be seen from the center of the Earth; $T+t$, $T'+t'$ the *Greenwich* times of the first and last contact, seen from the place of observation; l, l' the *true* longitudes of Venus and the Sun at the time T; ϕ the horizontal parallax of Venus, and f, g, &c. as before; then, if a and b are substituted for the coefficients of parallax (Art. 361.), the *apparent* difference of longitude at the time T

$$=l-l'+a\,(\phi-\pi);$$

therefore, the *apparent* difference of longitude at the time $T'+t$,

$$=l-l'+a\,.\,(\phi-\pi)+ft.$$

Similarly, the apparent latitude of Venus at the time $T+t$

$$=\lambda+b\,.\,(\phi-\pi)+gt.$$

Therefore, since the transit appears to begin at the time $T+t$, if c be the sum of the semi-diameters of the Sun and Venus,

$$c^2=(l-l'+a\,.\,\overline{\phi-\pi}+ft)^2+(\lambda+b\,.\,\overline{\phi-\pi}+gt)^2$$

$$=(l-l')^2+\lambda^2+2\,.\,(a\,.\,\overline{l-l'}+b\lambda)\,.\,(\phi-\pi)+2\,.\{f\,.\,(l-l')+g\lambda\}\,t;$$

neglecting the squares and products of the very small quantities t and $\phi-\pi$. But, since l, l', λ, are the longitudes and latitudes at the time the transit begins to a spectator in the Earth's center,

$$c^2=(l-l')^2+\lambda^2;$$

$$\therefore\ \{a.(l-l')+b\lambda\}.(\phi-\pi)+\{f.(l-l')+g\lambda\}\,t=0;$$

$$\therefore\ t=-\frac{a.(l-l')+b\lambda}{f.(l-l')+g\lambda}.(\phi-\pi)$$

$$=\alpha.(\phi-\pi),\text{ by substitution.}$$

Therefore, the Greenwich time of the first contact being seen at the place of observation

$$=T+\alpha.(\phi-\pi).$$

If β be the corresponding quantity to α for the time T', the time of the last contact

$$=T'+\beta.(\phi-\pi);$$

therefore, the whole duration (D)

$$=T'-T+(\beta-\alpha)\cdot(\phi-\pi).$$

If D' be the duration observed at any other place, and β', α' corresponding values of β and α,

$$D'=T'-T+(\beta'-\alpha').(\phi-\pi);$$

$$\therefore\ D'-D=(\alpha-\alpha'-\beta+\beta').(\phi-\pi);$$

$$\therefore\ \phi-\pi=\frac{D'-D}{\alpha-\alpha'-\beta+\beta'}.$$

Now, $\phi:\pi::$ the distance of the Sun from the Earth : the distance of Venus; therefore $\phi-\pi:\pi::$ the distance of Venus from the Sun : the distance of Venus from the Earth. Let this ratio, which is known, be that of 1 : e,

$$\therefore\ \pi=e.(\phi-\pi)=e.\frac{D'-D}{\alpha-\alpha'-\beta+\beta'}.$$

The transit of 1769 was observed in various places, and the most probable value of π, deduced from the different times of duration, was found to be $8''.5$.

379. The transits of Mercury, which occur oftener than those of Venus, may be used in the same manner for determining the Sun's parallax; but the observations are subject to greater errors, and consequently, no great dependance can be placed on the results.

◆

CHAP. XVIII.

TERRESTRIAL LONGITUDE.

380. If PZp, $PZ'p$ (fig. 74.) be the meridians of two places, whose zeniths are Z and Z', and PSp the declination circle passing through the Sun; ZPZ', the difference of longitude of the places $= ZPS - Z'PS =$ the difference of the apparent times at the two places, expressed in degrees. In the same manner it may be shewn, if S be the mean Sun or the first point of Aries, that the difference of longitude of two places $=$ the difference of the mean or sidereal times at the same places, expressed in degrees. Therefore, the problem of finding the difference of longitude reduces itself to finding the difference of the reckoned times, whether solar, sidereal, or mean-solar, for the same instant of absolute time, at the places for which the difference of longitude is required.

If PZp be the meridian of Greenwich, which the English take for their *first meridian*, ZPZ' is the west longitude of Z'; therefore, the longitude of any place $=$ the time at Greenwich $-$ the time at that place. If the time at the place of observation is greater than the time at Greenwich, the longitude of the place is measured east of Greenwich.

381. *To find the longitude of a place by a chronometer.*

The simplest method of finding the longitude is by a chronometer which can be depended on. If a person with such a chronometer, which points out the time under the meridian of Greenwich, were to travel to a place under a different meridian, the chronometer would continue to denote the Greenwich time; therefore, finding the time under the second meridian, by some of the methods of Chap. VI, and comparing it with the time shewn by the watch, the difference will be the longitude of the place of observation in time.

On account of the perfection with which these instruments are now made, this method is extremely valuable; but it is obvious that its value must depend in a great measure upon our having it in our power to ascertain if the watch goes uniformly, and upon our being able to determine and allow for any errors in its rate of going; that is, upon our being able to find at *any* place, by independent methods, the time under the first meridian, or the longitude of the place of observation.

382. The difference of longitude of two places may be determined astronomically, either by observing some phænomenon which is seen from both places at the same instant, such as a lunar eclipse, an eclipse of one of Jupiter's satellites, signals made on the Earth by the extinguishing of lamps, or the explosion of gunpowder; or else, by observing phænomena, not themselves happening for both places at the same instant, but from which the times of happening of some other phænomenon, which is independent of the places of observation, and therefore the same for both, may be deduced. Such phænomena are solar eclipses, occultations of fixed stars by the Moon, the transit of the Moon over the meridian, and the observed distance of the Moon from the Sun, or a fixed star.

383. *Longitude by lunar eclipses.*

A lunar eclipse, which is visible to half the Earth, would afford an excellent method of determining the longitude of places, if it could be observed with more accuracy; because, the eclipse happening for both places at the same instant of absolute time, the difference of the reckoned times at two places would give their difference of longitude. But, on account of the difficulty of determining the separation of the shadow and penumbra, two observers of an eclipse at the same place may differ more than 2′ in noting the time of contact; therefore, the longitude determined from the observed times of contact may be erroneous to the amount of 4′ of time, which produces an error of 1° of longitude. The error may be diminished, by observing at the two places the times when the shadow reaches known spots on the Moon's disk, determining the longitude from each pair of observations of the same spot, and taking the mean for the true longitude.

This method, even if it were more accurate, must be of little use at sea, on account of the comparative rarity with which lunar eclipses recur.

384. *Longitude by the eclipses of Jupiter's satellites.*

Although not so accurate as some of the succeeding methods, yet, on account of the frequency with which the eclipses recur, and the easiness of the observations, this method of finding the longitude is often used. In order to facilitate its application, the times at which the eclipses occur at Greenwich are given in the Nautical Almanack, by comparing which, with the observed times at any other place, the longitude of the place may be determined. The method cannot be used at sea on account of the difficulty of directing a telescope properly on ship-board, which; as yet, has been found insurmountable.

385. *Longitude by signals.*

Of all methods of determining small differences of longitude, perhaps the most accurate is that by signals made on the Earth's surface. A rocket, fired from an elevated spot on a clear night, may be seen distinctly with a telescope at a distance of twenty or thirty miles; therefore, by observing the times at which the same explosion is seen at two places, the difference of longitude of the places may be determined. If the places whose difference of longitude is required, are too distant to observe the same signal, a series of such signals must be made and noted by observers placed at intermediate places, from whence two adjacent signals can be seen. Here, as in other cases where accuracy is required, we must not be contented with the longitude determined from one observation, but take the mean of several longitudes determined by numerous observations.

The times at which the explosions are to take place, must be previously agreed on by the observers, and the telescopes must be properly directed during the day-time. Great attention also must be paid to the rate of the chronometers, and to the determination of the time at the places of observation, which is done by corresponding heights of the Sun during the preceding day, and also of a fixed star near the time of observation.

386. *Longitude by the transit of the Moon over the meridian.*

Of the second class of methods by which the longitude is determined, the simplest is by observing the increase of the Moon's right ascension in the interval between the transit of the Moon over the first meridian, and over the meridian whose longitude is required.

Let T be the time by the clock when the Moon is on the meridian of any place, t the time of transit of a known fixed star, $24+x$ the interval between two successive transits of the same star; then $24+x : T-t :: 360^0$: the difference of right ascension of the Moon and star at the instant the Moon was on the meridian, to which, adding the known right ascension of the star, the right ascension of the Moon (A) when on the meridian is determined. Now, the Moon's right ascension when on the meridian of Greenwich is given in the Nautical Almanack for every day of the year, from whence the daily increment of right ascension may be determined; let, therefore, a be the Moon's right ascension when on the meridian of Greenwich, E the increment of right ascension in the time between two successive transits over the same meridian; then, considering the change of right ascension uniform,

$$E : a-A :: 360^0 : \text{the required longitude.}$$

387. A method similar in principle to the preceding, but much more subject to error, is, by calculating the hour angle of

the Moon from an observed altitude, an approximate declination and the latitude of the place; then, the Moon's right ascension = the Sun's right ascension + the Sun's hour angle − the Moon's hour angle is known. Having found the right ascension of the Moon, it may be compared with the right ascension when on the meridian of Greenwich, and the difference will be the increase of the Moon's right ascension, whilst by the diurnal motion she describes an angle = the longitude of the place + the Moon's hour angle: this angle may be found exactly as in the preceding Article, and subtracting from it the Moon's hour angle, we find the longitude of the place.

388. *To find the longitude by the observed distance of the Moon from the Sun, or from a fixed star.*

Let A, a, δ be the *true* altitudes of the Sun and Moon, and the true distance of their centers, respectively; A', a', δ' the *observed* values of the same quantities; then, if P and R be the Moon's parallax and refraction, p and r the Sun's, calculated from the apparent altitudes A' and a' by the formulæ of Chap. ix.

$$A = A' - R + P, \quad a = a' - r + p.$$

Let s and m (fig. 75.) be the apparent places of the centers of the Sun and Moon, S and M the true places, Z the zenith, and let x = the angle SZM; then, from the triangle SZM,

$$\cos\delta = \sin A \, . \sin a + \cos A \, . \cos a \, . \cos x$$

$$= \sin A \sin a + \cos A \, . \cos a \, . (2 \cos^2 \tfrac{1}{2} x - 1)$$

$$= - \cos (A + a) + 2 \cos A \, . \cos a \cos^2 \tfrac{1}{2} x.$$

But $\cos\delta = 1 - 2 \sin^2 \frac{1}{2} \delta$, $\cos (A + a) = 2 \cos^2 \frac{1}{2} (A + a) - 1$;

$$\therefore \sin^2 \tfrac{1}{2} \delta = \cos^2 \tfrac{1}{2} (A + a) - \cos A \, . \cos a \, . \cos^2 \tfrac{1}{2} x.$$

And from the triangle Zsm,

$$\cos^2 \tfrac{1}{2} x = \frac{\sin \tfrac{1}{2} . (Zm + Zs + ms) . \sin \tfrac{1}{2} . (Zm + Zs - ms)}{\sin Zm . \sin Zs}$$

$$= \frac{\cos \tfrac{1}{2} (A' + a' - \delta') . \cos \tfrac{1}{2} . (A' + a' + \delta')}{\cos A' . \cos a'};$$

$$\therefore \sin^2 \tfrac{1}{2} \delta = \cos^2 \tfrac{1}{2} (A + a) -$$

$$\frac{\cos A . \cos a}{\cos A' . \cos a'} . \cos \tfrac{1}{2} (A' + a' - \delta') . \cos \tfrac{1}{2} (A' + a' + \delta').$$

Let the subsidiary angle ϕ be determined from the equation

$$\sin \phi = \sqrt{\left\{ \frac{\cos A . \cos a}{\cos A' . \cos a'} . \frac{\cos \tfrac{1}{2} (A' + a' - \delta') . \cos \tfrac{1}{2} (A' + a' + \delta')}{\cos^2 \tfrac{1}{2} (A + a)} \right\}};$$

$$\therefore \sin^2 \tfrac{1}{2} \delta = \cos^2 \tfrac{1}{2} (A + a) - \cos^2 \tfrac{1}{2} (A + a) . \sin^2 \phi;$$

$$\therefore \sin \tfrac{1}{2} \delta = \cos \tfrac{1}{2} (A + a) . \cos \phi,$$

from which equation δ, the true distance of the centers of the Sun and Moon, may be found.

The true distances of the Sun and Moon, and the corresponding Greenwich times, are given in the Nautical Almanack for every three hours, from whence, by taking the proportional part, the Greenwich time, when the distance is δ, may be found, and subtracting this from the time at the place of observation, we obtain the longitude of the place of observation in time.

389. Among the various ways by which the Moon's true distance (δ) is found, from the apparent distance and the apparent altitudes of the Sun and Moon, the preceding, by *Borda*, is considered the best. The following method is also one of great simplicity, as the latter part of the operation is performed by simple addition and subtraction, without requiring logarithms;

$$\because \cos\delta = \sin A\,.\sin a + \cos A\,.\cos a\,.\cos x\,;$$

$$\therefore \cos\delta = \sin A\,.\sin a + \cos A\,.\cos a\,(1 - 2\sin^2 \tfrac{1}{2}x)$$

$$= \cos(A - a) - 2\cos A\,.\cos a\,.\sin^2 \tfrac{1}{2}x,$$

$$= \cos(A - a) - \frac{2\cos A.\cos a}{\cos A'\cos a'}\,.\sin\tfrac{1}{2}\,.(A' + \delta' - a')\,.\sin\tfrac{1}{2}(a' + \delta' - A').$$

Let $\cos\theta = \frac{1}{2}\,\dfrac{\cos A\,.\cos a}{\cos A'\,\cos a'}\,;$

$$\therefore \cos\delta = \cos(A - a) - 4\cos\theta.\,\sin\tfrac{1}{2}(A' + \delta' - a').\,\sin\tfrac{1}{2}(a' + \delta' - A')$$

$$= \cos(A - a) + 2\cos\theta\,.\,\{\cos\delta' - \cos(A' - a')\}$$

$$= \cos(A - a) + \cos(\theta + \delta')$$

$$+ \cos(\theta - \delta') - \cos(\theta + A' - a') - \cos(\theta + a' - A')\,;$$

$$\therefore \text{ver-sin}\,\delta = \text{ver-sin}\,(A - a) + \text{ver-sin}\,(\delta' + \theta) + \text{ver-sin}\,(\delta' - \theta)$$

$$- \text{ver-sin}\,(A' - a' + \theta) - \text{ver-sin}\,(A' - a' - \theta).$$

390. The method of finding the longitude by the Moon's distance, is the only one which is of much use at sea, as it requires no other instrument than the sextant, and the observations are such as can be very frequently made. There ought to be three persons to observe at the same time the two altitudes, and the apparent distances; but where there is only one observer, it will be sufficient to observe the one altitude, then the distance, and then the other altitude, and immediately to repeat the operation of observing the altitudes, and from these observed altitudes, and the known intervals between the observations, to find, by taking the proportional parts, the altitude which would have been observed at the instant the distance between the centers was measured.

391. *To find the longitude from an observed occultation of a fixed star by the Moon.*

When this method is used, the longitude is supposed known, very nearly. Let θ be the approximate longitude in time, $\theta + t$ the true longitude, and let T be the reckoned time of the occultation at the place of observation; therefore, $T + \theta + t$ is the corresponding Greenwich time. Let the Moon's true latitude and longitude, taken out of the tables for the time $T + \theta$, be corrected for parallax, and let l and λ be the resulting apparent longitude and latitude of the Moon at the Greenwich time $T + \theta$, and f and g the horary increments of the same quantities; also, let l' and λ' be the latitude and longitude of the star; then, at the time of contact,

the difference of longitude $= l - l' + ft$,

. . . . latitude $= \lambda - \lambda' + gt$;

therefore, if μ be the semi-diameter of the Moon,

$$\mu^2 = (l - l' + ft)^2 \cos^2 \lambda + (\lambda - \lambda' + gt)^2$$

$$= (l - l')^2 \cos^2 \lambda + (\lambda - \lambda')^2 + 2\,\{f.(l - l').\cos^2 \lambda + g.(\lambda - \lambda')\}\,t,$$

neglecting the square of t;

$$\therefore\ t = \frac{\mu^2 - (\lambda - \lambda')^2 - (l - l')^2 \cos^2 \lambda}{2.\{g.(\lambda - \lambda') + f.(l - l')\cos^2 \lambda\}},$$

from whence the correct longitude $\theta + t$ is known.

392. The preceding solution has been obtained upon the supposition that the lunar tables are correct; but as this is not often the case, the values of λ and l will be subject to slight errors, let these be dl and $d\lambda$; then, we have

$$\mu^2 = (l + dl - l' + ft)^2 . \cos^2 \lambda + (\lambda + d\lambda - \lambda' + gt)^2$$

$$= (l - l') \cos^2 \lambda + (\lambda - \lambda')^2 + 2\,\{f.(l - l').\cos^2 l + g(\lambda - \lambda')\}\,t$$

$$+ 2\,(l - l') \cos^2 \lambda\,.\,dl + 2\,(\lambda - \lambda')\,d\lambda,$$

neglecting the squares and products of dl, $d\lambda$, and t.

This equation may be put under the form

$$A + Bt + Cdl + Dd\lambda = 0,$$

and if the same occultation be observed at two other places, whose longitudes are known, we shall be furnished with two more simple equations of the form

$$A' + C'dl + Dd\lambda = 0,$$

and from these three equations the values of t, dl, and $d\lambda$, may be found.

◆

APPENDIX.

APPENDIX (I).

THE PROJECTION OF THE SPHERE.

1. THE *Projection of the Sphere* is a perspective representation of the circles of the sphere on a plane given in position.

This Plane, called the *Primitive Plane,* generally passes through the center of the sphere, and if lines be drawn from the eye to all points of the circle to be represented, these will, by their intersection with the primitive, form the projection.

In the *Orthographic Projection,* the eye is supposed to be infinitely distant, so that the lines of vision are all parallel to each other, and perpendicular to the primitive plane.

2. Any line inclined to a plane : its orthographic projection :: 1 : cosine of inclination.

Let AB (fig. 76.) be any line, ab its projection on the plane DE; therefore Aa, Ba are perpendicular to ab. Draw AC parallel to ab, then $AC = ab$,

$$\text{and } AB : AC :: 1 : \cos BAC,$$

$$\text{or } AB : ab :: 1 : \cos \text{ inclination of } AB.$$

3. COR. 1. Hence the projection of a circle is an ellipse, whose axis-major equals the diameter of the circle, and whose axis-minor : axis-major :: cosine inclination : 1.

Let AB (fig. 77.) be that diameter which is parallel to the plane of projection, ECF the diameter perpendicular to AB, *PN* any ordinate parallel to *EC*, *ab*, *ec*, *pn* the projections of these lines.

$$\text{Then } PN : pn :: 1 : \cos \text{inclination}$$
$$:: EC : ec,$$

which is the property of an eclipse whose major axis *ab* equals *AB*, and minor axis (*ec*) : *ab* :: cos inclination : 1.

4. Cor. 2. The projection of a circle parallel to the plane of projection is an equal circle, and the projection of a circle perpendicular to the plane of projection is a straight line = diameter of the circle.

5. In the *Stereographic* projection, the eye is placed in the surface of a sphere at the extremity of a diameter, drawn perpendicular to the plane of projection which passes through the center of the sphere.

Let *P* (fig. 78.) be the place of the eye at the extremity of the diameter *Pp*, *HRO* the plane of projection dividing the sphere into two equal hemispheres; then, it is evident from the mode of projecting, that any point in the hemisphere *HpO* will be projected on the plane of the circle *HRO*, and any point in the hemisphere *HPO* will lie in that plane produced without the sphere. As the point to be projected lies nearer to *P*, its projection will be farther off, and the projection of points infinitely near *P*, will be infinitely distant. The projection of a great circle as *PHpO* passing through *P*, is the straight line *HO*, its intersection with the primitive, produced infinitely in opposite directions.

6. The stereographic projection of any circle not passing through the pole is a circle.

Retaining the preceding construction, let *AQB* (fig. 78.) be any circle, the plane of which is perpendicular to *PpO*, and let the cone formed by drawing lines from *P* to all points of *AQB*, cut the primitive in the curve *aqb*; then *aqb* is the projection of *AQB* and *ab* of *AB*. Through any point *N* in *aqb*, draw the plane *eqf*, parallel to *AQB*; and therefore perpendicular to *PpO*, cutting *aqb* in the line *Nq*; then since the section *eqf* is parallel to *AQB*, the circular base of the cone, *eqf* is a circle; and since *eqf* and *aqb* are perpendicular to the plane *PpO*, therefore *Nq* their intersection, is perpendicular to *PpO*; and therefore to the lines *ab* and *ef* in that plane.

But since *ef* is parallel to *AB*, therefore the angle $aqf = PAB = BPT$, (*PT* being a tangent at *P*) $= abf$, and the vertical angles at *N* are equal; therefore the triangle *aNe*, *bNf* are similar; therefore

$$aN : eN :: fN : bN;$$

$$\therefore\ aN . bN = eN . fN = qN^2;$$

therefore the curve *aqb* is a circle.

7. To find the radius and center of the stereographic projection of a given circle.

Constructing as in the preceding article, let *R* be the radius of *AQB*, *r* of its projection, $PC = a$, and $D =$ distance of the center of the projection from *C*.

$$\text{Let } Ap = 2\alpha,\quad Bp = 2\beta.$$

Then $r = \frac{1}{2} . (Cb - Ca) = \frac{1}{2} . (a \tan\beta - a \tan\alpha)$

$$= \frac{1}{2} a . \left(\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha}\right)$$

$$= \frac{1}{2} a \frac{(\sin\beta\cos\alpha - \sin\alpha\cos\beta)}{\cos\beta . \cos\alpha}$$

$$= \frac{1}{2} \frac{a \sin(\beta - \alpha)}{\cos\beta . \cos\alpha} = \frac{1}{2} \frac{R}{\cos\beta . \cos\alpha} \ldots (1).$$

And $D = Ca + \frac{1}{2} . (Cb - Ca) = \frac{1}{2} . (Cb + Ca)$

$$= \frac{1}{2} . (a \tan\beta + a \tan\alpha)$$

$$= \frac{1}{2} a \frac{\sin(\beta + \alpha)}{\cos\beta . \cos\alpha} \ldots\ldots\ldots (2).$$

8. **Cor. 1.** If the circle to be projected be parallel to the plane of the primitive, $\beta = -\alpha$;

$$\therefore r = \frac{R}{2\cos^2\alpha}, \quad D=0;$$

therefore C is the center of projection.

9. **Cor. 2.** If the circle be a great circle $2\beta - 2\alpha = 180$;

$\therefore \beta = 90^0 + \alpha$; $\therefore \cos\beta = -\sin\alpha$;

$$\therefore r = -\tfrac{1}{2}\cdot\frac{R}{\cos\alpha\sin\alpha} = -\frac{R}{\sin 2\alpha} = -R\operatorname{cosec} 2\alpha,$$

$$D = -\tfrac{1}{2}a\cdot\frac{\sin(90+2\alpha)}{\sin\alpha\,.\,\cos\alpha} = -\frac{D}{2}\,.\,a\frac{\cos 2\alpha}{\sin 2\alpha} = -a\cot 2\alpha.$$

But $2\alpha = 90^0 -$ the inclination of the circle to the plane of the primitive;

$$\therefore r = -R \text{ sect inclination},$$

$$D = -a \text{ tan inclination}.$$

When r and D are negative, the center of the stereographic projection lies on the other side of C.

10. The angle between any lines on the surface of the sphere is the same as that between their stereographic projections.

Let, as before, (fig. 78.) P be the place of the eye, PMp a great circle passing through P, M any point in this circle, m its projection. Let a plane touching the sphere at M cut the plane PMp in the line MT, and the primitive plane in the line TUV. Then, since this plane touches the sphere, the tangents to all curves, on the surface of the sphere passing through M, will lie in it, and MT will be a tangent to pM at M.

Let MU, MV be tangents to any other curves passing through M. Join mU, MV; then these are the projections of MU, MV, and therefore tangents to the projected curves; therefore UMV is the angle contained between two curves on the surface, and UmV the angle between their projections. Now since the angle

$$TMm = MpP = 90 - MPp = PmC = TmM;$$

$$\therefore TM = Tm;$$

and since MTV touches the sphere, it is perpendicular to CM, and therefore to the plane PMp, passing through CM; but the plane of the primitive is perpendicular to this plane; therefore their intersection TV, is perpendicular to it, and therefore to MT and Tm;

$$\therefore MU^2 = TU^2 + TM^2 = TU^2 + Tm^2 = Um^2;$$

$\therefore MU = Um$. But $TM = Tm$, and TU is common;

therefore the angle $TMU = TmU$; similarly, $TMV = TmV$; therefore the difference $UMV = UmV$.

11. The direction in which a ship sails, called the rhumb-line, makes a constant angle with the meridian. If, therefore, the Earth be stereographically projected on the equator, since the meridians are projected into right lines passing through the center, the projection of the rhumb-line will cut them all at the same angle, and is therefore the equiangular spiral.

Gnomonic Projection.

12. In the Gnomonic Projection the eye is placed in the center of the sphere, and the projection is made on a plane touching the sphere.

Every small circle in the gnomonic projection is projected in a conic section. For every small circle is the base of a cone, whose vertex is the eye, and the section of this cone by the plane of projection will be a circle, ellipse, parabola, or hyperbola.

Let GAH (fig. 79.) be the plane of projection, touching the sphere in A, P the center of the sphere, BPC a plane parallel to GH, DE any small circle, AD being greater than AE; then if AD be less than 90^0, the projection is an ellipse, axis-major GH; if $AD=90^0$, GH is parallel to the side of the cone PB, and the section is a parabola; if AD be greater than 90^0, D falls below B, and GH meets the side DP of the cone produced backwards; therefore the section is an hyperbola, axis-major GH.

The axes of these projections are easily deduced, and if $AD=2\alpha$, $AE=2\beta$, $AP=a$.

$$\text{The axis-major} = a\,.\,\frac{\sin 2(\alpha-\beta)}{\cos 2\alpha\,.\,\cos 2\beta},$$

$$\text{the axis-minor} = 2a\,.\,\frac{\sin 2(\alpha-\beta)}{\sqrt{\cos 2\alpha\,.\,\cos\beta}}.$$

13. Not to dwell longer on particular cases, any problem of this kind may be solved by the following general method.

Take the plane of projection for the plane of xy, and let the axis of z pass through the pole P; let

$$F(pqr)=0,\ f(pqr)=0,$$

be the equations of the curve to be projected, p, q, r being co-ordinates of any point Q in it. Then the equations of the line PQ are

$$py = qx \ \ldots\ldots\ldots\ (1),$$

$$p(z-c) = \overline{r-c}\,.\,x \ \ldots\ldots\ (2),$$

c being the distance of P from the plane xy. Eliminating therefore p, q, r between these equations, and the equation of the curve, we have the equation of the conical surface, which is the locus of PQ, from which, by making $z=0$, we get the equation of the section made by the plane of xy, or the equation of the projection.

14. Ex. Let the origin of the co-ordinates be the center of a sphere, and the curve to be projected a circular section of

the sphere made by a plane perpendicular to the plane of zx. Then, the equations of this circle are, if a be the radius of the sphere,

$mp+nr=mn$...(3), the equation of the cutting plane,

$p^2+q^2+r^2=a^2$....(4), ... sphere.

From equations (1), (2), (3), we easily get

$$p=\frac{n.\overline{m-c}.x}{mx+\overline{z-c}.n},$$

$$q=\frac{n.\overline{m-c}.y}{mx+\overline{z-c}.n},$$

$$r=\frac{cmx+mn.\overline{z-c}}{mx+\overline{z-c}.n}.$$

Substituting these values in (4), we have the equation of the conic surface

$$n^2.(m-c)^2.(x^2+y^2)+\{cmx+mn(z-c)\}^2=\{mx+n.(z-c)\}^2a^2;$$

therefore, the equation of the projection is

$$n^2.(m-c)^2.(x^2+y^2)+m^2c^2.(x-n)^2=(mx-nc)^2a^2.$$

This equation becomes the equation of the Stereographic, or Orthographic Projection, according as c is a, or infinite.

15. All perspective representations of the sphere distort those parts which are not projected near the center of the primitive plane. In maps of small portions of the Earth, this is not of much consequence, as the middle of the map may be always taken for the center of the projection, but when extensive tracts are to be represented, the countries near the edge of the map must differ considerably from their true proportion. Lagrange, in the Berlin Memoirs for 1779, has investigated modes of drawing charts which coincide most nearly with the countries to be represented, and other methods may be seen in the

Encyclopedie Methodique, Art, Cartes. Here it is only necessary to notice *Mercator's Projection*, according to which sea-charts are generally drawn.

In Mercator's projection, the meridians are represented by parallel lines, and consequently, the degrees of longitude, instead of decreasing as on the globe in proportion to the cosines of latitude, are all equal, and the degrees of latitude in the projection are increased to preserve the same relation between the degrees of latitude and longitude.

16. To find the length of an arc of the meridian on Mercator's projection, supposing the Earth spherical.

A degree of the meridian in any latitude l (which is the same as a degree of longitude at the equator) : a degree of longitude :: 1 : cos l; therefore, a degree of the projection in latitude l varies as sec l, since the degrees of longitude are all equal. Let ds be an arc of the meridian in latitude l, corresponding to an increment of latitude dl, then, at the equator $ds=dl$; and sect $l = 1$;

$$\therefore\ ds : dl :: \text{sect}\, l : 1;$$

$$\therefore\ ds = \frac{dl}{\cos l} = \frac{\cos l \,.\, dl}{\cos^2 l} = \frac{d \,.\, \sin l}{1 - \sin^2 l};$$

$$\therefore\ s = \tfrac{1}{2}\,\text{hyp log}\,\frac{1+\sin l}{1-\sin l} = \text{hyp log tan}\,(45 + \tfrac{1}{2}\,l),$$

which is the length of an arc of the meridian between the equator and latitude l.

17. To find the length of an arc of a meridian in Mercator's projection, the Earth being supposed spheroidal.

Retaining the same notation, let 1 be equatoreal radius, e the eccentricity of the elliptic meridian, r the radius of curvature of the meridian in latitude l, x the radius of a parallel to the equator.

Then, a degree of the meridian in latitude l : a degree of longitude :: $r : x$; therefore, the degree of the meridian in the projection $\propto \frac{r}{x}$, since the degrees of longitude are all equal.

Let ds be an arc of the projection corresponding to dl; then, at the equator, $ds = dl \times$ radius of curvature of the meridian at the equator, $= (1-e^2) \, . \, dl$, and

$$\frac{r}{x} = \frac{1-e^2}{(1-e^2\sin^2 l)^{\frac{3}{2}}} \cdot \frac{(1-e^2\sin^2 l)^{\frac{1}{2}}}{\cos l} \quad \text{(Art. 23, 24.)}$$

$$= \frac{1-e^2}{\cos l \, . \, (1-e^2\sin^2 l)};$$

$$\therefore \; ds : \overline{1-e^2} \, . \, dl :: \frac{1-e^2}{\cos l \, . \, (1-e^2\sin^2 l)} : 1-e^2;$$

$$\therefore \; ds = \frac{\overline{1-e^2} \, . \, dl}{\cos l \, . \, (1-e^2\sin^2 l)} = \frac{\overline{1-e^2} \, . \, \cos l \, . \, dl}{\cos^2 l \, . \, (1-e^2\sin^2 l)}$$

$$= \frac{\overline{1-e^2} \, . \, d \, . \sin l}{(1-\sin^2 l)\,(1-e^2\sin^2 l)}$$

$$= \frac{d \, . \sin l}{1-\sin^2 l} - \frac{e^2 \, . \, d \, . \sin l}{1-e^2\sin^2 l};$$

$$\therefore \; s = \tfrac{1}{2} \,\text{hyp log}\, \frac{1+\sin l}{1-\sin l} - e \, . \, \tfrac{1}{2} \, . \,\text{hyp log}\, \frac{1+e\sin l}{1-e\sin l}$$

$$= \text{hyp log tan}\,(45 + \tfrac{1}{2} l) - e \, . \, \{e\sin l + \tfrac{1}{3} e^3 \sin^3 l + \&c.\}$$

$$= S + e^2 \sin l, \text{ nearly,}$$

since e is very small, where S is the projection, on the hypothesis of the Earth's being spherical.

18. The defect of Mercator's projection consists in the degrees towards the pole being much elongated, but this is

compensated for by the rhumb-line, in the direction of which a ship sails being represented in sea-charts by a straight line.

APPENDIX (II).

ON DIALLING.

1. If a rectilinear gnomon be placed parallel to the Earth's axis, the angular motion of its shadow on a plane parallel to the equator will be uniform. If the gnomon coincided with the Earth's axis, its shadow would revolve uniformly on any plane parallel to the equator, because the Sun's diurnal motion may be considered uniform during the same day; and since the distance between the gnomon and the Earth's axis subtends an angle, the greatest value of which does not exceed $8''.57$, they may be considered coincident; therefore the shadow, cast by a gnomon placed parallel to the Earth's axis, will revolve uniformly on a plane parallel to the equator, at the rate or 15^0 an hour.

2. Def. A line drawn from the center of the dial plane in the direction of the Earth's axis, is called the *Style*. A declination circle drawn through the style perpendicular to the dial plane intersects it in a line called the *Substyle*. The *Style's height* is the angle between the style and substyle. *Hour lines* are lines drawn from the center of the dial coincident with the shadow of the style, at the interval of each successive hour. Hence the twelve o'clock hour line, which coincides with the shadow of the style when the Sun is on the meridian, is the intersection of the plane of the meridian and the dial. Dials are denominated, according to the position of their planes. *Equatoreal, Horizontal,* &c.

3. *To construct an* Equatoreal *Dial.*

From the definitions, and Art. 1, it appears that in this dial the style is perpendicular to the dial plane, and the hour lines are drawn at intervals of 15^0. Since the dial is parallel to the equator, its inclination to the horizon = the co-latitude.

4. *To construct a* horizontal *Dial.*

Let $NHLS$ (fig. 80.) be the plane of the dial, OP the style, NO the 12 o'clock hour line, which, since the meridian passes through PO perpendicular to the horizon, is also the substyle; therefore PON, the style's height, = the altitude of the pole, or latitude of the place. At n hours after noon, let OH, to the east of ON, be the hour line; then, drawing the declination circle POH through OH, $NPH = n \,.\, 15^0$;

$$\text{therefore } \sin \text{lat.} = \tan NH \,.\, \cot n\, 15^0,$$

$$\text{or } \tan NOH = \sin \text{lat.} \tan n\, 15^0,$$

which determines the hour line NH; therefore giving n the values 1, 2, 3, &c. the hour lines corresponding to one, two, and three o'clock may be drawn, and drawing similarly situated lines on the west side of ON, these are the hour lines corresponding to 11 o'clock, 10 o'clock, &c.

If OL be the last hour line on any day, OL = the Sun's south azimuth at setting, and, (Art. 131.)

$$\cos OL = \frac{\sin \text{Sun's dec.}}{\cos \text{lat.}};$$

therefore the least value of OL, or the last hour line on the dial is found by the equation

$$\cos OL = \frac{\sin 23^0\ 28''}{\cos \text{lat.}}.$$

5. *To construct a* vertical south *dial.*

Let ZHN (fig. 81.) be the dial plane coinciding with the prime vertical, Op the style directed towards the south pole,

ZpN the meridian; therefore ON is the 12 o'clock hour line, and also, (since the meridian is perpendicular to the prime vertical) the substyle; and $NOp = ZOP =$ the co-latitude. Let OH be the hour line n hours from noon; then, as before,

$$\sin Np = \tan NH \,.\, \cot NpH;$$

$$\text{therefore } \tan NH = \cos \text{lat.} \tan n\, 15^0,$$

from which, by giving n the different values 1, 2, 3, &c. the different hour lines may be drawn, observing that the hour lines before noon are to the west of ON, and the others to the east.

If OL be the last hour line on any day, the Sun is then in the plane of the dial, and $pH =$ the Sun's south polar distance $= 90^0 + \delta$; but

$$\sin pH = \cos Np \,.\, \cos NL;$$

$$\text{therefore } \sin \delta = \sin \text{lat.} \cos NL,$$

which equation determines NL. The greatest value of NL corresponds to $\delta = 0$, in which case $NL = 90^0$, and it is useless to graduate the dial beyond this.

6. *To construct a* vertical declining *dial inclined at a given angle to the meridian.*

Let ZSN (fig. 81.) be the dial's plane passing through the vertical ZN, and inclined at a given angle pNH (e) to the meridian PZp; therefore ON is the 12 o'clock hour line. Let Op be the style, OS the substyle, which falls amongst the morning or evening hour lines, according as the dial inclines east or west. Then, since $Np =$ the co-latitude, we have, in the right-angled triangle pSN,

$\sin pS = \cos \text{lat.} \sin e$, which gives pS, the style's height.

$\cos e = \tan \text{lat.} \tan SN$, ... SN, the place of the substyle,

$\sin \text{lat.} = \cot e \,.\, \cot NpS$, ... NpS (ϕ).

Let now OH be the hour line n hours from 12 o'clock;

$$\therefore\ SpH = \phi \mp n\,15^0,$$

$\sim$ or $+$ according as the substyle and hour line are on the same or different sides of ON. Then in the right-angled triangle SpH, we have

$$\tan SH = \sin pS\,.\,\tan(\phi \mp n\,15^0),$$

which determines SH, and giving n the values 1, 2, 3, &c. all the hour lines may be determined.

If OL be the last hour line on any day, the Sun is then in the plane of the dial, and we have given $pL = 90^0 +$ Sun's declination, $pN =$ the co-latitude, and $pNS =$ the Sun's azimuth, to find SL.

7. *To construct an* East *or* West *dial.*

Let POp (fig. 82.) be the dial which coincides with the plane of the meridian, O its center, Pp in the plane of the dial parallel to the Earth's axis, ASB the style which is parallel to Op, and formed by the edge of a rectangular lamina, AD, passing through Pp, perpendicular to the dial plane; then, since the plane AD is perpendicular to the meridian, CD is the six o'clock hour line; and since AB is parallel to the plane of the dial, all the other hour lines are parallel to AB or Pp. Let aHb be the hour line n hours from six o'clock, OSH a plane perpendicular to Pp or AB, intersecting the dial plane in the line OH; then, since the Sun revolves uniformly round AB, $OSH = n\,.\,15^0$, and SOH is a right angle;

$$\therefore\ OH = OS\,.\tan n\,15^0,$$

which determines the hour line ab, and giving n different values, all the hour lines may be drawn.

As the Sun only shines on the dial during half the day, if the dial fronts the east, it points out the time from sun-rise to noon, or if the dial fronts the west, from noon to night.

8. *To construct a* general dial, *inclined at any given angles to the meridian and horizon.*

Let SON (fig. 83.) be the plane of the dial inclined at an angle HNR (N) to the meridian PZH, and at an angle RAN, (A) to the horizon; therefore, ON is the 12 o'clock hour line. Let OP be the style passing through the pole P, OS the substyle; therefore, $PSN = 90°$. But $ARN = 90^0$; therefore, by Napier's rules,

$$\cos A = \sin N . \cos RN,$$

whence RN, and, therefore, PN = the latitude $+ RN$ are known. And from the right-angled triangle PSN,

$\cos PN = \cot SPN . \cot N$, which gives SPN (ϕ),

$\cos N = \tan SN . \cot PN$.......... OS, the substyle,

$\sin SP = \sin N . \sin PN$..........$SP$, the style's height ($h$).

At n hours after 12, let OH be the hour line; therefore,

$$SPH = \phi - n . 15^0;$$

$$\therefore \sin h = \cot (\phi - n\ 15^0) . \tan SH,$$

from which equation, by giving n the values 1, 2, 3, all the hour lines may be drawn.

9. The time pointed out by a dial is apparent time, but, in consequence of the atmospheric refraction, which elevates the Sun in a vertical circle, the shadow only points out this time exactly at noon, the dial being fast all the morning, and slow in the evening.

The gnomon is generally formed by a triangular lamina of metal, the sides of which are, the style, substyle, and a perpendicular let fall from one on the other. If the thickness of this lamina is considerable, the shadow of the western edge points out the morning hours, and the eastern the evening, and the hour lines must be drawn for each separately, so as to form two half dials with different centers.

A solar dial may be made use of for determining the time by means of the Moon's shadow, if we know the time when the Moon is on the meridian. For the time pointed out by the shadow is the apparent time from the Moon's being on the meridian, nearly; therefore, adding the time on the dial to the time of the Moon's transit, the solar time is determined. The method is not accurate, as the Moon moves through a little more than 15^0 in a solar hour.

10. To find the curve traced out by the extremity of the shadow of a vertical gnomon on a horizontal plane.

Let AB (fig. 84.) be the gnomon, AP the shadow, AN the direction of the meridian shadow. Draw PN perpendicular to AN, and let $AN=x$, $PN=y$, $AB=a$, $l=$ the latitude of the place, $\delta=$ the Sun's declination. Then

$$\text{sin Sun's altitude} = \frac{AB}{BP} = \frac{a}{\sqrt{x^2+y^2+a^2}},$$

$$\text{cos Sun's altitude} = \frac{AP}{BP} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+a^2}},$$

$$\text{cos Sun's azimuth} = \frac{AN}{AP} = \frac{x}{\sqrt{x^2+y^2}}.$$

But $\sin\delta = \cos l \,.\, \text{cos altitude} \,.\, \text{cos azimuth} + \sin l \,.\, \text{sin altitude}$;

$$\therefore\ \sin\delta = \cos l \,.\, \frac{x}{\sqrt{x^2+y^2+a^2}} + \frac{\sin l \,.\, a}{\sqrt{x^2+y^2+a^2}};$$

$$\therefore\ x^2+y^2+a^2 = \frac{(\cos l \,.\, x + \sin l \,.\, a)^2}{\sin^2\delta};$$

$$\therefore\ y^2 = \frac{(\cos^2 l - \sin^2\delta) \,.\, x^2 + 2a\sin l\cos l x + (\sin^2 l - \sin^2\delta) \,.\, a^2}{\sin^2\delta}.$$

Cor. If $\cos l = \sin\delta$, or $l = 90^0 - \delta$ the curve is a parabola,

$\cos l > \sin\delta$, or $l < 90^0 - \delta$ .... a hyperbola,

$\cos l < \sin\delta$, or $l > 90^0 - \delta$ .... an ellipse.

APPENDIX (III).

INTERPOLATIONS.

1. IT is often requisite in Astronomy, to determine the general relation between two quantities, from a number of corresponding values; as from a number of zenith distances taken at given intervals, to determine the zenith distance for any intermediate time. This problem, which is the same as that of finding the equation to a curve passing through a number of given points, is plainly indeterminate, but if the corresponding values are numerous and near to each other, the general relation, within the limits of the given quantities, may be found to a great degree of accuracy.

2. Let a, b, c, &c. be values of y, corresponding to α, β, γ, values of x, and let

$$y = Aa + Bb + Cc + \&c.,$$

where A, B, C, are functions of x to be determined. The second member of this equation ought to reduce itself to a, when $x = \alpha$, which will be the case if all the coefficients but A vanish, when $x = \alpha$, and A become unity; but, in order that B, C, &c. may vanish when $x = \alpha$, they must contain the factor $\overline{x - \alpha}$, and as the same reasoning applies to B and C, all but B must contain $\overline{x - \beta}$, and all but C, $\overline{x - \gamma}\ldots$

Let, therefore, $A = A'.\overline{x-\beta}.\overline{x-\gamma}\ldots$ which ought to equal 1, when $x = \alpha$;

$$\therefore\ A'.(\alpha-\beta).(\alpha-\gamma)\ldots = 1;\ \therefore\ A' = \frac{1}{(\alpha-\beta).(\alpha-\gamma)\ldots};$$

$$\therefore\ A = \frac{(x-\beta).(x-\gamma)\ldots}{(\alpha-\beta).(\alpha-\gamma)\ldots};$$

$$\therefore\ y = \frac{(x-\beta).(x-\gamma)\ldots}{(\alpha-\beta).(\alpha-\gamma)\ldots}\,a + \frac{(x-\alpha).(x-\gamma)\ldots}{(\beta-\alpha).(\beta-\gamma)\ldots}\,b + \&c.$$

3. When the values of x are taken at equal intervals, as generally happens in Astronomy, a more simple formula will suffice.

Let a, b, c, &c. be values of y corresponding to the values of x, 0, 1, 2, 3, &c.

$$\text{Let } y = a + Bx + C.x.\frac{\overline{x-1}}{2} + D.x.\frac{\overline{x-1}.\overline{x-2}}{2.3} + \text{\&c.}$$

(B, C, &c. being constants);

$$\therefore\ a = a,$$

$$b = a + B;\quad \therefore\ B = b - a,$$

$$c = a + 2B + C;\quad \therefore\ C = c - 2b + a,$$

$$d = a + 3B + 3C + D;\quad \therefore\ D = d - 3c + 3b - a,$$

&c. = &c.

Here it is obvious that B, C, &c. are the successive differences of a, b, c, &c.; therefore, calling these differences d', d'', &c.

$$y = a + d'.x + d''.x.\frac{x-1}{2} + d'''.x.\frac{x-1}{2}.\frac{\overline{x-2}}{3} + \text{\&c.}$$

In general, the quantities d', d'', &c. diminish very fast, and it will not often be necessary to proceed farther than d'''.

4. The following example will shew some of the different uses that may be made of the formulæ of interpolation.

Ex. Given the Sun's declinations at 12 o'clock on the 19th, 20th, 21st, 22d of June,

$$= 23^{0}\ 26'\ 39'',\quad 23^{0}\ 27'\ 23'',\quad 23^{0}\ 27'\ 42'',\quad 23^{0}\ 27'\ 37'',$$

respectively; to find the declination on the 20th, at 6 o'clock.

$$\begin{array}{llllllll}
a & = 23^0\,26'\,39''; & & 23^\circ\,27'\,23''; & & 23^0\,27'\,42'; & & 23^\circ\,27'\,37'', \\
d' & = & 44'' & & 19'' & & -5'', & \\
d'' & = & & -25'' & & -24'', & & \\
d''' & = & & & 1'', & & &
\end{array}$$

and $x = 1\frac{1}{4} = \frac{5}{4}$,

therefore, the required declination

$$= 23^0\,26'\,39'' + \frac{5}{4}.44'' - \frac{5}{4}.\frac{1}{4}.\frac{25''}{2} - \frac{5}{4}.\frac{1}{4}.\frac{3}{4}.\frac{1''}{2.3} + \&c.$$

$$= 23^0\,26'\,46''.$$

The declination (y) at any time x

$$= 23^0\,26'\,39'' + x\,.\,44'' - x\,.\,\frac{x-1}{2}\,.\,25''.$$

Since the declination increases and then diminishes, it admits at some intermediate time of a maximum, the time being determined by the equation

$$44'' - \frac{2.x-1}{2}.25'' = 0;$$

$$\therefore\ x = \frac{113}{50} = \frac{226}{100} = 2.26^d = 2^d\,6^h\,14'\,24'',$$

and substituting for x, $\frac{113}{50}$,

$$y = 23^0\,27'\,42''.8.$$

Therefore the Sun's greatest declination (or the obliquity) falls on June 21,

$$\text{at } 6^h\,14'\,24'', \text{ and} = 23^0\,27'\,42''.$$

5. In astronomical investigations, it is sometimes required to determine the values of several unknown quantities from simple equations, which are more in number than the quantities to be determined. The coefficients of these equations are generally determined by observation, and if the observations are correct, the values of the unknown quantities deduced from some of the equations ought, when substituted, to satisfy the remainder; but as this is never the case, the unknown quantities ought to be so determined, as to be least subject to error. If the equations be combined, so that the factor of one of the unknown quantities, as x, is the greatest possible, and the coefficients of the others, y, z, &c. the least possible, x will be determined most accurately, since any errors in the values of y, z, &c. will have less influence on the value of x, being multiplied by the least factors, and divided by the greatest coefficient of x. In the same manner equations may be formed, to give the values of y, z, &c. most accurately, and from all these equations, which will be the same in number as the unknown quantities, x, y, z, &c. may be found. The equations for determining x, y, &c. may be formed by changing the signs, so that all the coefficients of x may be the same, and then adding, and proceeding in the same way by y, z, &c.

Ex. Let the equations be

$$3- x+ y-2z=0,$$

$$5-3x-2y+5z=0,$$

$$21-4x- y-4z=0,$$

$$14+ x-3y-3z=0,$$

changing the sign of the last equation, and adding

$$15-gx+ y+ 2z=0,$$

similarly for y, $37-5x-7y=0$,

for z, $33- x- y=14z=0$.

From these equations, $x=2.486$,

$$y=3.517,$$

$$z=1.928,$$

which three values substituted in the original equations give, instead of 0, the results

$$0.175,\ 0.148,\ -0.173,\ 0.151.$$

Second method.

$$\begin{aligned} \text{Let } m &+ax+by+cz+\&\text{c.}=0, \\ m' &+a'x+b'y+c'z+\&\text{c.}=0, \\ m'' &+a''x-b''y+c''z+\&\text{c.}=0, \\ \&\text{c.} &\ldots\ldots\ldots\ldots\ldots\ldots\ldots=0, \end{aligned}$$

be the equations, multiply the first by a, the second by a', and so on; then, by addition,

$$ma+m'a'+\&\text{c.}+(a^2+a'^2+\&\text{c.})x+(ab+a'b'+\&\text{c.})y+(ac+a'c+\&\text{c.})z+\&$$

Similarly,

$$(mb-m'b'+\&\text{c.})+(ab+a'b'+\&\text{c.})x+(b^2+b'^2+\&\text{c.})y+(bc+b'c'+\&\text{c.})z+\&$$

$$(mc+m'c'+\&\text{c.})+(ac+a'c'+\&\text{c.})x+(bc+b'c'+\&\text{c.})y+(c^2+c'^2+\&\text{c.})z+\&$$

$$\&\text{c.}+\&\text{c.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

By this means as many equations are formed as there are unknown quantities, and from them x, y, z, &c. may be determined.

The method applied to the example in the preceding article, gives the reduced equations

$$\begin{aligned} -\ 88+27x+\ 6y&=0 \\ -\ 70+\ 6x+15z&=0 \\ -107+\ y\ +54z&=0. \end{aligned}$$

From whence $x = 2.470$; $y = 3.551$; $z = 1.916$.

The above mode of reducing the linear equations, which is called the *Method of Least Squares*, was invented by Gauss, and is proved by him in his *Theoria motus Corporum Cœlestium*, to give the most probable values of the unknown quantities. The proof is not sufficiently elementary to be given here. See also Laplace, *Theorie des Probabilites*, Chap. IV.

◆

ERRATA.

Page 29, l. 24, *for* Ophinci *read* Ophiuci.
31, l. 19, *omit* be.
32, l. 17, *for* of *read* from.
44, l. 2, *for* $\cos \epsilon$ *read* $\cos 2\epsilon$.
55, l. 2, *for* $\cos a$ *read* $\sin a$.
— l. 17, *for* $\cos l$ *read* $\sin l$.
63, l. 6, *for* $7' \, 0''.1$ *read* $7' \, 0''.31$.
— l. 19, *for* 15° *read* 360°.
72, l. 14, 15, *transpose* Z, Z', *and also* S, S'.
100, l. 20, *for* Z *read* Z'.
113, l. 6, *for* $\sin. ZP . \sin PS$ *read* $\cos ZP . \sin. PS$.
120, l. 26, *for* $9''.05$ *read* $9''.65$.
137, l. 13, 14, *for* $10''.25$, *read* $20''.25$.
225, l. 2, *for* β *read* λ.
261, l. 14, *for* $\frac{1}{8}$ *read* $\frac{1}{3}$.
263, l. 22, 21, 25, *for* OL *read* NL.
264, l. 13, *for* $\sin pH$ *read* $\cos pL$.
266, l. 13, *for* ON *read* OS.

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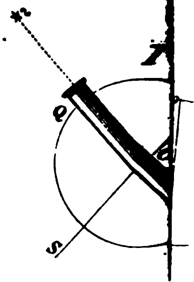

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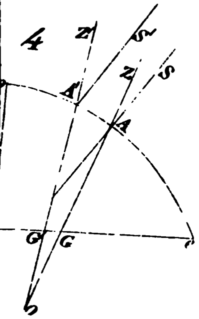

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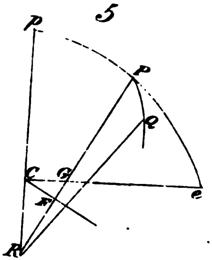

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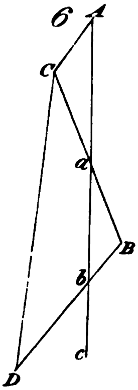

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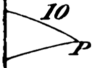

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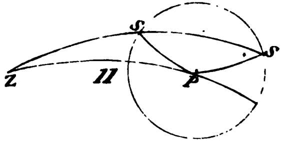

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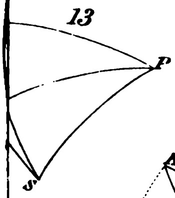

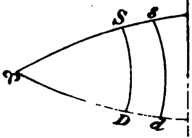

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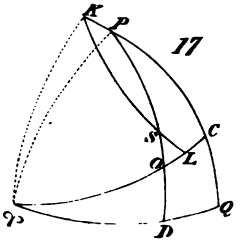

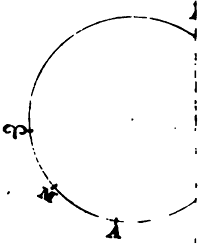

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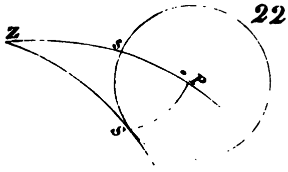

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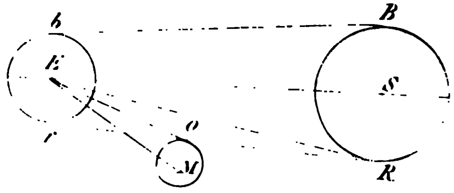

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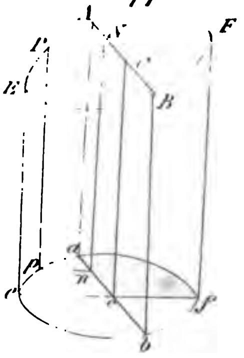

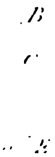

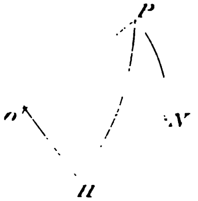

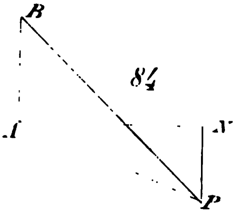

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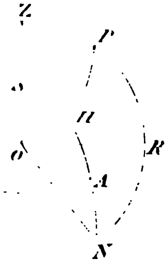

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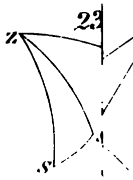

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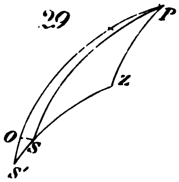

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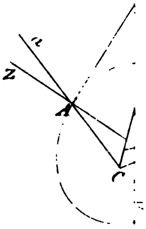

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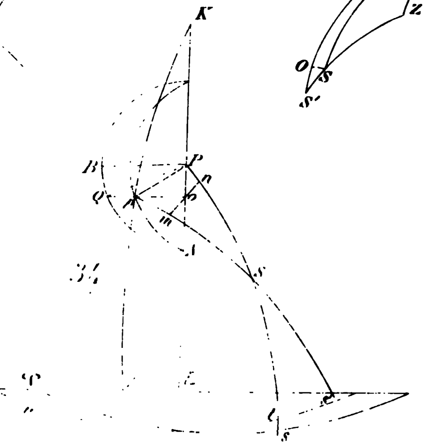

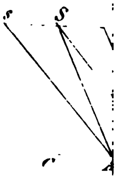

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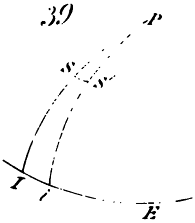

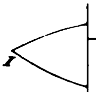

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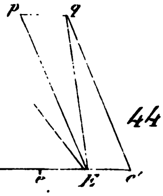

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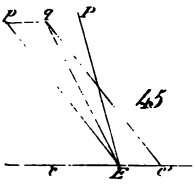

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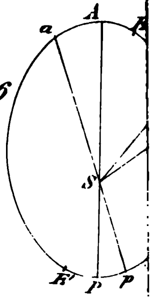

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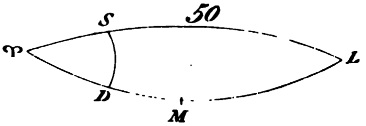

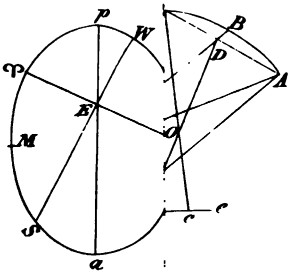

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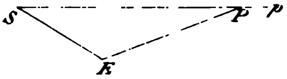

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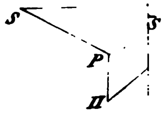

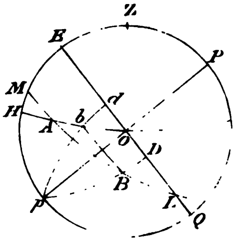

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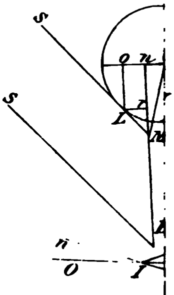

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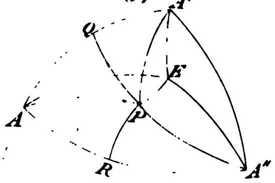

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